

Bounded complexes on Deligne-Mumford stacks

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Abstract

Laszlo and Olsson constructed Grothendieck's six operations for unbounded complexes on Artin stacks under an assumption of finite cohomological dimension. In this article we construct a formalism of six operations for bounded complexes on Deligne-Mumford stacks without this assumption. We develop a theory of gluing of pseudo-functors, which allows us to prove base-change theorems in derived categories.

Introduction

Let S be a regular scheme of dimension ≤ 1 , ℓ a prime number invertible on S . In [LO08a], [LO08b], Laszlo and Olsson construct a formalism of Grothendieck's six operations for unbounded complexes of ℓ -power torsion and ℓ -adic lisse-étale sheaves on finite type Artin S -stacks, under the assumption that the ℓ -cohomological dimension of every finite type S -scheme is finite. Note that a field does not satisfy this assumption in general. In this article, we construct a formalism of six operations for *bounded* complexes of torsion and ℓ -adic étale sheaves on finite type Deligne-Mumford S -stacks without the above assumption. This formalism generalizes that of [SGA 4 $\frac{1}{2}$, Th. finitude] and [Eke90, 6.3].

Let Λ be a Gorenstein ring of dimension 0 annihilated by an integer m invertible on S . We construct, in § 2, for every morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of Deligne-Mumford S -stacks of finite type, functors

$$(0.0.1) \quad f^*, Rf^!: D_c^b(\mathcal{Y}, \Lambda) \rightarrow D_c^b(\mathcal{X}, \Lambda),$$

and, if f is of prime to m inertia (1.12), functors

$$(0.0.2) \quad Rf_*, Rf_!: D_c^b(\mathcal{X}, \Lambda) \rightarrow D_c^b(\mathcal{Y}, \Lambda).$$

As in [LO08a], $Rf^!$ and $Rf_!$ are constructed by duality and the key point is the gluing of the dualizing complex.

If f is of prime to m inertia, under the additional assumptions that f is separated and that \mathcal{X} and \mathcal{Y} are of finite inertia, which are satisfied if \mathcal{X} and \mathcal{Y} are separated, $Rf^!$ and $Rf_!$ are restrictions of more general functors

$$Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda), \quad Rf_!: D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda),$$

which we construct in § 1 by applying Nagata compactification [CLO09] to the coarse spaces. This more direct approach allows us to construct the base change isomorphism (1.18) in the derived category, which was constructed on the level of sheaves in [LO08a, § 5]. We also construct the support-forgetting morphism $Rf_! \Rightarrow Rf_*$, which was constructed in several cases by Olsson [Ols08, 5.1, 5.17].

The construction does not depend on the choice of the compactification. In [SGA 4, XVII] Deligne checked this for schemes by gluing *two* fibered categories. The case of stacks is more complicated, because f is isomorphic to a composition of *three* morphisms (1.10). We set up the necessary framework of gluing in an appendix (§ 7).

In § 3, we develop an ℓ -adic formalism for a general topos. We closely follow the methods of [LO08b]. In § 4, we apply this formalism to Deligne-Mumford stacks and construct analogues of (0.0.1) and (0.0.2)

with Λ replaced by a complete discrete valuation ring \mathcal{O} with fraction field E of characteristic 0 and residue field F of characteristic ℓ . Again, we are able to construct the base change isomorphism (4.20) and the support-forgetting morphism. In § 5, we show that the restriction on the inertia of f disappears when we pass from \mathcal{O} to E .

Our formalism is used in [IZ09, § 2] to give a generalization of Laumon's theorem on Euler-Poincaré characteristics. In *loc. cit.*, the cases of E -coefficients and F -coefficients are treated separately. In § 6, we use Brauer theory to prove that the case of F -coefficients follows from the case of E -coefficients.

We have largely ignored logical problems that are usually solved by choosing a universe \mathfrak{U} . By a regular scheme, we mean a Noetherian regular scheme.

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1 Operations on $D(\mathcal{X}, \Lambda)$

1.1. Let Λ be a commutative ring with unity. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of toposes. The exact functor $f^*: \text{Mod}(\mathcal{Y}, \Lambda) \rightarrow \text{Mod}(\mathcal{X}, \Lambda)$ induces a triangulated functor

$$f^*: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda).$$

The left exact functor $f_*: \text{Mod}(\mathcal{X}, \Lambda) \rightarrow \text{Mod}(\mathcal{Y}, \Lambda)$ has a right derived functor [KS06, 18.6]

$$Rf_*: D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda).$$

For $M \in D(\mathcal{X}, \Lambda)$ and $N \in D(\mathcal{Y}, \Lambda)$, the projection formula map

$$(1.1.1) \quad N \otimes_{\Lambda}^L Rf_* M \rightarrow Rf_*(f^* N \otimes_{\Lambda}^L M)$$

is adjoint to the composition

$$f^*(N \otimes_{\Lambda}^L Rf_* M) \xrightarrow{\sim} f^* N \otimes_{\Lambda}^L f^* Rf_* M \rightarrow f^* N \otimes_{\Lambda}^L M,$$

where the second map is induced by the adjunction $f^* Rf_* M \rightarrow M$.

If \mathcal{X} is algebraic, \mathcal{Y} is locally coherent [SGA 4, VI 2.3] and f is coherent [SGA 4, VI 3.1], then $R^q f_*$ commutes with small filtrant inductive limits for all q [SGA 4, VI 5.1]. Thus, in this case, f_* -acyclic sheaves on \mathcal{X} are stable under small filtrant inductive limits. If, moreover, f_* is of finite cohomological dimension, then Rf_* commutes with small direct sums [KS06, 14.3.4 (ii)].

1.2. Let S be a quasi-separated scheme. All algebraic S -spaces are assumed to be quasi-separated. We use a more general notion of Deligne-Mumford stacks than that used in [LMB00]. By a *Deligne-Mumford S -stack*, we mean an S -stack \mathcal{X} such that the diagonal $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable, quasi-compact and quasi-separated, and such that there exists an algebraic S -space X and an étale surjective morphism $X \rightarrow \mathcal{X}$. The base scheme does not play much role, in the sense that if $S' \rightarrow S$ is a morphism of quasi-separated schemes, the 2-category of Deligne-Mumford S' -stacks is 2-equivalent [GR09, 1.2.10 (i)] to the 2-category of Deligne-Mumford S -stacks over S' . By a *Deligne-Mumford stack*, we mean a Deligne-Mumford $\text{Spec}(\mathbb{Z})$ -stack.

Let \mathcal{X} be a Deligne-Mumford S -stack. The 2-category of Deligne-Mumford S -stacks representable over \mathcal{X} is 2-equivalent to the 1-category $\text{Rep}(\mathcal{X})$ obtained by identifying isomorphic 1-cells. The objects of $\text{Rep}(\mathcal{X})$ are pairs (\mathcal{Y}, f) , where \mathcal{Y} is a Deligne-Mumford S -stack and $f: \mathcal{Y} \rightarrow \mathcal{X}$ is a representable morphism. A morphism of $\text{Rep}(\mathcal{X})$ from (\mathcal{Y}, f) to (\mathcal{Z}, g) is an equivalence class of pairs (h, α) , where $h: \mathcal{Y} \rightarrow \mathcal{Z}$ is a morphism of Deligne-Mumford S -stacks and $\alpha: f \rightarrow gh$ is a 2-cell. In such a pair, h is necessarily a representable étale morphism. Two such pairs (h, α) and (i, β) are equivalent if there

exists a 2-cell $\gamma: h \Rightarrow i$ such that $\beta = (g\gamma)\alpha$, or, in other words, that β is equal to the composition of the following 2-cells

$$\begin{array}{ccc} & i & \\ \gamma \Uparrow & & \\ \mathcal{Y} & \xrightarrow{h} & \mathcal{Z} \\ \alpha \nearrow & & \downarrow g \\ & f & \mathcal{X} \end{array}$$

We define the étale site $\text{Et}(\mathcal{X})$ of \mathcal{X} to be the full subcategory $\text{Et}(\mathcal{X})$ of $\text{Rep}(\mathcal{X})$ consisting of Deligne-Mumford S -stacks representable and étale over \mathcal{X} , endowed with the étale topology. The category $\text{Et}(\mathcal{X})$ admits finite projective limits. The corresponding topos \mathcal{X}_{et} is algebraic. The full subcategory of $\text{Et}(\mathcal{X})$ consisting of affine S -schemes étale over \mathcal{X} , endowed with the étale topology, gives the same topos [SGA 4, III 4.1]. If \mathcal{X} is quasi-compact, then \mathcal{X}_{et} is coherent.

Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford S -stacks. The functor

$$f^{-1}: \text{Et}(\mathcal{Y}) \rightarrow \text{Et}(\mathcal{X}), \quad \mathcal{U} \mapsto \mathcal{U} \times_{\mathcal{Y}} \mathcal{X}$$

is left exact and continuous [SGA 4, III 1.6]. Hence it induces a morphism of toposes $(f_*, f^*): \mathcal{X}_{\text{et}} \rightarrow \mathcal{Y}_{\text{et}}$ [SGA 4, III 1.3]. If f is quasi-compact, then this morphism of toposes is coherent. If $g: \mathcal{X} \rightarrow \mathcal{Y}$ is also a morphism and $\alpha: f \Rightarrow g$ is a 2-cell, α induces a natural transformation $g^{-1} \Rightarrow f^{-1}$, and hence a 2-cell $(f_*, f^*) \Rightarrow (g_*, g^*)$.

A surjective smooth morphism of Deligne-Mumford stacks is of cohomological descent, because étale locally it has a section.

1.3. Let \mathcal{X} be a topos and U be an object of \mathcal{X} . Let $\mathcal{U} = \mathcal{X}/U$ and consider the morphism of toposes $j: \mathcal{U} \rightarrow \mathcal{X}$. The restriction functor $j^*: \text{Mod}(\mathcal{X}, \Lambda) \rightarrow \text{Mod}(\mathcal{U}, \Lambda)$ admits a left adjoint

$$j_!: \text{Mod}(\mathcal{U}, \Lambda) \rightarrow \text{Mod}(\mathcal{X}, \Lambda)$$

[SGA 4, IV 11.3.2]. We denote $j_! \Lambda_{\mathcal{U}}$ by $\Lambda_{\mathcal{U}, \mathcal{X}}$. The functor $j_!$ is exact and induces a triangulated functor

$$j_!: D(\mathcal{U}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda).$$

For $M \in D(\mathcal{U}, \Lambda)$, $N \in D(\mathcal{X}, \Lambda)$, the adjunction morphisms

$$(1.3.1) \quad R\text{Hom}_{\mathcal{X}}(j_! M, N) \rightarrow R\text{Hom}_{\mathcal{U}}(M, j^* N)$$

and

$$(1.3.2) \quad R\text{Hom}_{\mathcal{X}}(j_! M, N) \rightarrow Rj_* R\text{Hom}_{\mathcal{U}}(M, j^* N)$$

are isomorphisms. They are induced by

$$\text{Hom}_{\mathcal{X}}^{\bullet}(j_! M, N') \xrightarrow{\sim} \text{Hom}_{\mathcal{U}}^{\bullet}(M, j^* N'),$$

and

$$\mathcal{H}om_{\mathcal{X}}^{\bullet}(j_! M', N') \xrightarrow{\sim} j_* \mathcal{H}om_{\mathcal{U}}^{\bullet}(M', j^* N'),$$

where N' is homotopically injective and equipped with a quasi-isomorphism $N \rightarrow N'$, M' belongs to $\tilde{\mathcal{P}}_{\mathcal{U}}$ and is equipped with a quasi-isomorphism $M' \rightarrow M$. Here $\tilde{\mathcal{P}}_{\mathcal{U}}$ is the smallest full triangulated subcategory of $K(\text{Mod}(\mathcal{U}, \Lambda))$ closed under small direct sums and containing $K^-(\mathcal{P}_{\mathcal{U}})$, where $\mathcal{P}_{\mathcal{U}}$ is the full additive subcategory of $\text{Mod}(\mathcal{U}, \Lambda)$ consisting of flat sheaves. Note that j^* preserves homotopically injective complexes. The map (1.3.2) is adjoint to the composition

$$j^* R\text{Hom}_{\mathcal{X}}(j_! M, N) \xrightarrow{\sim} R\text{Hom}_{\mathcal{U}}(j^* j_! M, j^* N) \rightarrow R\text{Hom}_{\mathcal{U}}(M, j^* N),$$

where the second map is deduced from the adjunction $M \rightarrow j^* j_! M$.

The projection formula map

$$(1.3.3) \quad j_!(j^* N \otimes_{\Lambda}^L M) \rightarrow N \otimes_{\Lambda}^L j_! M,$$

adjoint to the composition

$$j^* N \otimes_{\Lambda}^L M \rightarrow j^* N \otimes_{\Lambda}^L j^* j_! M \simeq j^*(N \otimes_{\Lambda}^L j_! M),$$

where the first map is deduced from the adjunction $M \rightarrow j^* j_! M$, is an isomorphism. In fact, (1.3.3) is induced by the isomorphism of complexes [KS06, 18.2.5]

$$j_! \operatorname{tot}_{\oplus}(j^* N' \otimes_{\Lambda} M) \xrightarrow{\sim} \operatorname{tot}_{\oplus}(N' \otimes_{\Lambda} j_! M)$$

where N' belongs to $\tilde{\mathcal{P}}_{\mathcal{X}}$ and is equipped with a quasi-isomorphism $N' \rightarrow N$.

1.4. Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of toposes. Let $V = f^{-1}(U)$, $\mathcal{V} = \mathcal{Y}/V$ and consider the following 2-commutative square D of toposes

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{j'} & \mathcal{Y} \\ \downarrow g & & \downarrow f \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X} \end{array}$$

The base change map (7.18.2), natural transformation of functors $D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{U}, \Lambda)$

$$B_D: j^* Rf_* \Rightarrow Rg_* j'^*,$$

is a natural equivalence. By 7.21, the base change map (7.20.1) is a natural equivalence of functors $D(\mathcal{U}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$

$$(1.4.1) \quad A_D: j'_! g^* \Rightarrow f^* j_!,$$

and B_D^{-1} and A_D^{-1} induce by adjunction the same natural transformation of functors $D(\mathcal{V}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$

$$(1.4.2) \quad G_D: j_! Rg_* \Rightarrow Rf_* j'_!.$$

1.5. Let $j: \mathcal{U} \rightarrow \mathcal{X}$ be a representable étale morphism of Deligne-Mumford S -stacks. Then 1.3 applies to j . If j is an open immersion, the adjunction map $j^* Rj_* \Rightarrow \mathbf{1}_{D(\mathcal{U}, \Lambda)}$ is invertible and we have, by 7.22, a natural transformation of functors $D(\mathcal{U}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$

$$(1.5.1) \quad j_! \Rightarrow Rj_*,$$

compatible with composition of open immersions.

If

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{j'} & \mathcal{Y} \\ \downarrow g & & \downarrow f \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X} \end{array}$$

is a 2-Cartesian [GR09, 1.2.12] square of Deligne-Mumford S -stacks with j representable and étale, then 1.4 applies.

1.6. Let $i: \mathcal{Y} \rightarrow \mathcal{X}$ be a closed immersion of Deligne-Mumford S -stacks. Then

$$i_*: \operatorname{Mod}(\mathcal{Y}, \Lambda) \rightarrow \operatorname{Mod}(\mathcal{X}, \Lambda)$$

is exact. Define $i^! : \text{Mod}(\mathcal{X}, \Lambda) \rightarrow \text{Mod}(\mathcal{Y}, \Lambda)$ by $i^! \mathcal{F} = i^* \text{Ker}(\mathcal{F} \rightarrow j_* j^* \mathcal{F})$. Then $i^!$ is a right adjoint of i_* and thus is left exact and has a right derived functor $Ri^! : D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$. Let $j : \mathcal{U} \rightarrow \mathcal{X}$ be the complementary open immersion.

For any complex M of Λ -modules on \mathcal{X} , we have a natural short exact sequence

$$0 \rightarrow j_! j^* M \rightarrow M \rightarrow i_* i^* M \rightarrow 0,$$

hence a distinguished triangle in $D(\mathcal{X}, \Lambda)$

$$(1.6.1) \quad j_! j^* M \rightarrow M \rightarrow i_* i^* M \rightarrow .$$

For any complex N of injective Λ -modules on \mathcal{X} , we have a natural short exact sequence

$$0 \rightarrow i_* i^! N \rightarrow N \rightarrow j_* j^* N \rightarrow 0.$$

It follows that, for any $N \in D(\mathcal{X}, \Lambda)$, we have a distinguished triangle

$$(1.6.2) \quad i_* Ri^! N \rightarrow N \rightarrow Rj_* j^* N \rightarrow .$$

In fact, it suffices to take a quasi-isomorphism $N' \rightarrow N$, where N' is homotopically injective with injective components [KS06, 14.1.6, 14.1.7].

For $M \in D(\mathcal{X}, \Lambda)$, $N \in D(\mathcal{X}, \Lambda)$, the morphism

$$(1.6.3) \quad R\mathcal{H}om_{\mathcal{Y}}(i^* M, Ri^! N) \rightarrow Ri^! R\mathcal{H}om_{\mathcal{X}}(M, N),$$

adjoint to the composition

$$i_* R\mathcal{H}om_{\mathcal{Y}}(i^* M, Ri^! N) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{X}}(i_* i^* M, N) \rightarrow R\mathcal{H}om_{\mathcal{X}}(M, N),$$

where the second map is deduced from the adjunction $M \rightarrow i_* i^* M$, is an isomorphism. In fact, we have an isomorphism of distinguished triangles

$$\begin{array}{ccccccc} R\mathcal{H}om_{\mathcal{X}}(i_* i^* M, N) & \longrightarrow & R\mathcal{H}om_{\mathcal{X}}(M, N) & \longrightarrow & R\mathcal{H}om_{\mathcal{X}}(j_! j^* M, N) & \longrightarrow & \\ \downarrow & & \parallel & & \downarrow & & \\ i_* Ri^! R\mathcal{H}om_{\mathcal{X}}(M, N) & \longrightarrow & R\mathcal{H}om_{\mathcal{X}}(M, N) & \longrightarrow & Rj_* j^* R\mathcal{H}om_{\mathcal{X}}(M, N) & \longrightarrow & \end{array}$$

Proposition 1.7 (Smooth base change). *Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-Cartesian square of Deligne-Mumford S -stacks, $M \in D(\mathcal{X}, \Lambda)$. Assume either

- (a) *g is étale, or*
- (b) *g is smooth, f is quasi-compact, Λ is annihilated by an integer $m \in \mathbb{Z}$ invertible on S , and $M \in D^+(\mathcal{X}, \Lambda)$.*

Then the base change map

$$g^* Rf_* M \rightarrow Rf'_* h^* M$$

is an isomorphism.

Proof. This is standard. We give a proof for the sake of completeness.

Consider the diagram with 2-Cartesian square

$$\begin{array}{ccccc} Y'' & \xrightarrow{\beta} & Y' & \xrightarrow{g'} & Y \\ & & \downarrow \alpha' & & \downarrow \alpha \\ & & \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

where α and β are étale presentations with Y and Y'' disjoint unions of quasi-compact schemes. Since base change by α and $\alpha'\beta$ holds trivially, up to replacing g by $g'\beta$, we may assume that \mathcal{Y} and \mathcal{Y}' are quasi-compact schemes. Case (a) is then trivial. In case (b), take a quasi-compact étale presentation $\gamma: X \rightarrow \mathcal{X}$ with X an algebraic S -space. Let $X_\bullet = \text{cosk}_0 \gamma$ and let $\gamma_\bullet: X_\bullet \rightarrow \mathcal{X}$ be the projection. Then $X_n = (X/\mathcal{X})^{n+1}$ is an algebraic S -space (even if we had taken X to be an S -scheme). Take $X'_\bullet = \mathcal{X}' \times_{\mathcal{X}} X_\bullet$ (2-fiber product) and consider the square

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{h_\bullet} & X_\bullet \\ \downarrow \gamma'_\bullet & & \downarrow \gamma_\bullet \\ \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \end{array}$$

By cohomological descent, the adjunction $M \rightarrow R\gamma_{\bullet*} \gamma_\bullet^* M$ is an isomorphism. It follows that it suffices to show that the base change maps

$$g^* R(f\gamma_\bullet)_* M_\bullet \rightarrow R(f'\gamma'_\bullet)_* h^* M_\bullet, \quad h^* R\gamma_{\bullet*} M_\bullet \rightarrow R\gamma'_{\bullet*} h'_* M_\bullet$$

are isomorphisms, where $M_\bullet = \gamma_\bullet^* M$. For the second map, repeating the first reduction of this proof, we may assume that \mathcal{X} and \mathcal{X}' are quasi-compact schemes. Therefore, we are reduced to proving the theorem under the additional hypotheses that \mathcal{Y} , \mathcal{Y}' are quasi-compact schemes and \mathcal{X} is an algebraic S -space. In this case, take an étale presentation $X \rightarrow \mathcal{X}$ with X a quasi-compact scheme and repeat the preceding reduction. We may then assume that \mathcal{X} is also an S -scheme. In this case, the result is classical [SGA 4, XVI 1.2]. \square

Proposition 1.8. *Assume that Λ is annihilated by an integer invertible on S . Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a smooth morphism of Deligne-Mumford S -stacks, $M, L \in D(\mathcal{Y}, \Lambda)$. Then the map*

$$(1.8.1) \quad f^* R\mathcal{H}om_{\mathcal{Y}}(M, L) \rightarrow R\mathcal{H}om(f^* M, f^* L)$$

is an isomorphism.

Proof. The problem is local for the étale topology on \mathcal{X} and on \mathcal{Y} . We may assume that \mathcal{X} and \mathcal{Y} are quasi-compact schemes and f is separated. Then f is compactifiable and (1.8.1) becomes the inverse of the trivial duality [SGA 4, XVIII 3.1.12.2] via the isomorphism $f^*(d)[2d] \xrightarrow{\sim} Rf^!$ [SGA 4, XVIII 3.2.5], where d is the relative dimension of f . Note that the trivial duality holds in fact for unbounded complexes: the proof of (1.23.2) applies. \square

1.9. We say that a morphism f of Deligne-Mumford S -stacks is a *universal homeomorphism* if it is a homeomorphism and remains so after every 2-base change of Deligne-Mumford S -stacks. Note that we do not assume f to be representable. A universal homeomorphism is universally injective, hence separated. Unlike the case of schemes, a universal homeomorphism does not induce an equivalence of étale toposes in general.

Assume in the rest of this section that S is a Noetherian scheme.

Proposition 1.10. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford S -stacks of finite type and finite inertia. Then f is isomorphic to the composition of morphisms of Deligne-Mumford stacks*

$$\mathcal{X} \xrightarrow{\pi} \mathcal{X}' \xrightarrow{j} \mathcal{Z} \xrightarrow{p} \mathcal{Y},$$

where π is a proper homeomorphism, j is an open immersion, p is proper and representable. Moreover, if f is quasi-finite, we can take p to be finite.

Proof. Let $g: X \rightarrow Y$ be the morphism of coarse spaces [Con05, 1.1] associated to f . Then g is a separated morphism of algebraic S -spaces of finite type. Applying Nagata compactification [CLO09, 1.2.1] to g , we get $g = qk$, where $q: Z \rightarrow Y$ is a proper morphism of algebraic spaces and $k: X \rightarrow Z$ is an open immersion. It then suffices to take p to be the base change of q and j to be the base change of k . If f is quasi-finite, so is g , and it suffices to apply Zariski's Main Theorem [LMB00, 16.5] to g . \square

Let \mathcal{C} be a 2-category. The *inertia* of a 1-cell $f: X \rightarrow Y$ is defined to be

$$I_f = X \times_{\Delta_f, Y, \Delta_f} X.$$

The following is an immediate consequence of [IZ09, 2.11].

Lemma 1.11. *Let*

$$\begin{array}{ccccc} U' & \xrightarrow{\quad} & X' & & \\ & \searrow & \downarrow & \searrow & \\ & U & \xrightarrow{\quad} & X & \\ & \downarrow & \downarrow & \downarrow & \\ V' & \xrightarrow{\quad} & Y' & & \\ & \searrow & \downarrow & \searrow & \\ & V & \xrightarrow{\quad} & Y & \end{array}$$

be a 2-commutative cube in \mathcal{C} with 2-Cartesian bottom and top squares. Then the square

$$\begin{array}{ccc} I_{U'/V'} & \xrightarrow{\quad} & I_{X'/Y'} \\ \downarrow & & \downarrow \\ I_{U/V} & \xrightarrow{\quad} & I_{X/Y} \end{array}$$

is 2-Cartesian.

It follows that Deligne-Mumford S -stacks of finite inertia are stable under 2-fiber products.

1.12. Let m be an integer. Following [IZ09, 2.9], we say that a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of Deligne-Mumford S -stacks is *of prime to m inertia* if for every algebraically closed field Ω and every point $x \in \mathcal{X}(\Omega)$, the order of the group

$$\mathrm{Aut}_{\mathcal{X}_y}(x) \simeq \mathrm{Ker}(\mathrm{Aut}_{\mathcal{X}}(x) \rightarrow \mathrm{Aut}_{\mathcal{Y}}(y))$$

is prime to m , where $y \in \mathcal{Y}(\Omega)$ is the image of x under f . Note that morphisms of prime to m inertia are closed under composition and base change.

Proposition 1.13. *Let m be an integer, \mathcal{C} be the 2-category of Deligne-Mumford S -stacks of finite type and finite inertia whose 1-cells are the separated morphisms of prime to m inertia. Let \mathcal{X} be an object of \mathcal{C} , $\mathcal{A}_{\mathcal{X}}$ be the 2-faithful subcategory (7.1) of \mathcal{C}/\mathcal{X} whose 1-cells are the open immersions, $\mathcal{B}_{\mathcal{X}}$ be the 2-faithful subcategory of \mathcal{C}/\mathcal{X} whose 1-cells are the proper morphisms, \mathcal{D} be a 2-category. Then the 2-functor (7.3.3)*

$$(1.13.1) \quad \mathrm{PsFun}(\mathcal{C}/\mathcal{X}, \mathcal{D}) \rightarrow \mathrm{GD}_{\mathcal{A}_{\mathcal{X}}, \mathcal{B}_{\mathcal{X}}}(\mathcal{C}/\mathcal{X}, \mathcal{D})$$

is a 2-Fun($\mathrm{Ob}(\mathcal{C}/\mathcal{X}), \mathcal{D}$)-equivalence (7.1).

Proof. To simplify notations, let $\mathcal{C}' = \mathcal{C}/\mathcal{X}$, $\mathcal{A} = \mathcal{A}_{\mathcal{X}}$, $\mathcal{B} = \mathcal{B}_{\mathcal{X}}$ and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ be the 2-faithful subcategories of \mathcal{C}' whose 1-cells are respectively the quasi-finite, representable and quasi-finite, quasi-finite and proper, finite morphisms. Then the functor (1.13.1) is a composition of functors

$$\begin{aligned} \text{PsFun}(\mathcal{C}', \mathcal{D}) &\xrightarrow{E_1} \text{GD}_{\mathcal{A}_1, \mathcal{B}}(\mathcal{C}', \mathcal{D}) \xrightarrow{E_2} \text{GD}_{\mathcal{B}_1, \mathcal{A}_2, \mathcal{B}}(\mathcal{C}', \mathcal{D}) \\ &\xrightarrow{P_1} \text{GD}_{\mathcal{A}_2, \mathcal{B}}(\mathcal{C}', \mathcal{D}) \xrightarrow{E_3} \text{GD}_{\mathcal{A}, \mathcal{B}_2, \mathcal{B}}(\mathcal{C}', \mathcal{D}) \xrightarrow{P_2} \text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}', \mathcal{D}), \end{aligned}$$

where P_1 and P_2 are equivalences of categories by 7.15 since $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$, and E_1, E_2 and E_3 are equivalences by 7.16: E_1 and E_2 satisfy condition (i) of 7.16 by 1.10, E_3 satisfies (i) by Zariski's Main Theorem, E_2 satisfies (vi') since $\mathcal{A}_1 \cap \mathcal{B} = \mathcal{B}_1$ by definition, E_3 satisfies (vi') since $\mathcal{A}_2 \cap \mathcal{B} = \mathcal{B}_2$ by Zariski's Main Theorem. \square

Assume in the rest of this section that Λ is annihilated by an integer $m \neq 0$. Unless otherwise stated, we do not assume that m is invertible on S . We say that a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of Deligne-Mumford S -stacks is *S -proper*, if f is the 2-base change of some proper morphism $f_0: \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ of Deligne-Mumford S -stacks with \mathcal{Y}_0 locally of finite type over S . Recall that Chow's lemma [LMB00, 16.6] implies base change for S -proper morphisms and D^+ . For morphisms of prime to m inertia, this can be generalized to unbounded complexes as follows.

Lemma 1.14. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be an S -proper morphism of Deligne-Mumford S -stacks. Assume that f is of prime to m inertia and the fibers have dimension $\leq d$. Then f_* has cohomological dimension $\leq 2d$.*

Proof. Up to replacing \mathcal{Y} by an étale presentation and \mathcal{X} be the corresponding 2-pull back, we may assume that \mathcal{Y} is a separated scheme. Then f factorizes through the coarse space of \mathcal{X} , and we are reduced to two cases: (a) f is a universal homeomorphism; (b) f is representable. By proper base change for D^+ , we may assume that \mathcal{Y} is the spectrum of an algebraically closed field. In case (a), $\mathcal{X}^{\text{red}} = BG$, G of order prime to m . In this case, f_* can be identified with the functor of taking G -invariants, which is exact. Case (b) follows from [IZ09, 6.4]. We give a proof that does not make use of cohomology with proper support. We proceed by induction on d . By Chow's lemma [Knu71, IV.3.1], there exists $\pi: X \rightarrow \mathcal{X}$ proper and birational such that X is a scheme. For any $\mathcal{F} \in \text{Mod}(\mathcal{X}, \Lambda)$, complete the adjunction $\mathcal{F} \rightarrow R\pi_*\pi^*\mathcal{F}$ into a distinguished triangle

$$\mathcal{F} \rightarrow R\pi_*\pi^*\mathcal{F} \rightarrow M \rightarrow .$$

By the case of schemes of this lemma [SGA 4, X 4.3, 5.2], $Rf_*R\pi_*\pi^*\mathcal{F} \simeq R(f\pi)_*\pi^*\mathcal{F} \in D^{\leq 2d}$ and, for any i , the support of $\mathcal{H}^i M$ has dimension $\leq d - 1 - i/2$. By induction hypothesis, $Rf_*M \in D^{\leq 2d-2}$. Thus $Rf_*\mathcal{F} \in D^{\leq 2d}$. \square

Proposition 1.15. (i) (Proper base change) Let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-Cartesian square of Deligne-Mumford S -stacks where f is S -proper and of prime to m inertia, $M \in D(\mathcal{X}, \Lambda)$. Then the base change map

$$g^*Rf_*M \rightarrow Rf'_*h^*M$$

is an isomorphism.

(ii) (Projection formula) Let $f: X \rightarrow Y$ be an S -proper morphism of Deligne-Mumford S -stacks, $M \in D(\mathcal{X}, \Lambda)$, $N \in D(\mathcal{Y}, \Lambda)$. Assume that f is of prime to m inertia. Then the map (1.1.1)

$$N \otimes_{\Lambda}^L Rf_*M \rightarrow Rf_*(f^*N \otimes_{\Lambda}^L M)$$

is an isomorphism.

Proof. (i) We may assume \mathcal{Y} quasi-compact. By 1.14, we are then reduced to the known case when $M \in D^+$.

(ii) By (i), we may assume that \mathcal{Y} is the spectrum of a separably closed field. Since Rf_* commutes with small direct sums (1.1), we may assume $M \in D^-(\mathcal{X}, \Lambda)$, and N is represented by a complex bounded above with flat components. Then we may assume $M \in \text{Mod}(\mathcal{X}, \Lambda)$, and N is a flat Λ -module. Then N is a filtrant inductive limit of finite free Λ -modules. Since $R^q f_*$ commutes with such limits (1.1), we may assume that N is a finite free Λ -module. In this case (1.1.1) is obviously an isomorphism. \square

Proposition 1.16. *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}_0 & \xrightarrow{f_0} & \mathcal{Y}_0 \end{array}$$

be a 2-Cartesian square of Deligne-Mumford S -stacks where f_0 is a proper universal homeomorphism (1.9) of prime to m inertia, \mathcal{Y}_0 is locally of finite type over S . Let $M \in D(\mathcal{Y}, \Lambda)$. Then the adjunction map $M \rightarrow Rf_ f^* M$ is an isomorphism.*

Proof. By proper base change (1.15), we may assume that \mathcal{Y} is the spectrum of an algebraically closed field. Then $\mathcal{X}^{\text{red}} = BG$, G of order prime to m . In this case the assertion is trivial. \square

1.17. We use the gluing result 1.13 to give a first construction of $Rf_!$. Let \mathcal{C} , $\mathcal{A} = \mathcal{A}_S$, $\mathcal{B} = \mathcal{B}_S$ be as in 1.13, \mathcal{D} be the 2-category of triangulated categories. Define an object $(F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho)$ of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ (7.6) as follows. Let $F_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{D}$ be the pseudo-functor given by 1.5:

$$\mathcal{X} \mapsto D(\mathcal{X}, \Lambda), \quad j \mapsto j_!, \quad \alpha \mapsto \alpha_!,$$

and let $F_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{D}$ be the pseudo-functor given by

$$\mathcal{X} \mapsto D(\mathcal{X}, \Lambda), \quad p \mapsto Rp_*, \quad \alpha \mapsto R\alpha_*.$$

If f is a proper open immersion, the 2-cell (1.5.1) $f_! \Rightarrow Rf_*$ is invertible and let $\rho(f)$ be its inverse. If D is a 2-Cartesian square (7.3.1), let $G_D: i_! Rq_* \Rightarrow Rp_* j_!$ be the 2-cell as in (1.4.2). Then $i^* G_D$ can be identified with $\mathbf{1}_{Rq_*}$, hence is an isomorphism. Consider the complementary square

$$\begin{array}{ccc} Y - X & \xrightarrow{j'} & Y \\ \downarrow q' & & \downarrow p \\ W - Z & \xrightarrow{i'} & W \end{array}$$

The proper base change map

$$(1.17.1) \quad i'^* Rp_* \Rightarrow Rq'_* j'^*$$

is an isomorphism by 1.15 (i), hence $i'^* G_D$ is $0 \Rightarrow 0$. It follows that G_D is an isomorphism. Axioms (b), (b'), (c), (c') follow from 7.23. By 7.9 and 1.13, this defines a pseudo-functor $F: \mathcal{C} \rightarrow \mathcal{D}$. For any 1-cell of \mathcal{C} , namely, any separated morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of prime to m inertia of Deligne-Mumford S -stacks of finite type and finite inertia, we define

$$Rf_!: D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$$

to be $F(f)$.

The gluing formalism also enables us to construct the following natural transformations.

(i) *Support-forgetting map.* Let $F': \mathcal{C} \rightarrow \mathcal{D}$ be the pseudo-functor given by

$$\mathcal{X} \mapsto D(\mathcal{X}, \Lambda), \quad f \mapsto Rf_*, \quad \alpha \mapsto R\alpha_*.$$

Let $\epsilon_A: F|A \Rightarrow F'|A$ be the pseudo-natural transformation given by (1.5.1) and $\epsilon_B: F|B \Rightarrow F'|B$ be the identity. Then (ϵ_A, ϵ_B) is a 1-cell of $\text{GD}'_{A,B}(\mathcal{C}, \mathcal{D})$ by 7.23 (iv), thus defines a pseudo-natural transformation $F \Rightarrow F'$. For any 1-cell $f: \mathcal{X} \rightarrow \mathcal{Y}$ of \mathcal{C} , this defines a natural transformation of functors $D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$

$$f! \Rightarrow Rf_*,$$

which is a natural equivalence if f is proper.

(ii) *Base change isomorphism.* Let $g: \mathcal{X}' \rightarrow \mathcal{X}$ be a morphism of Deligne-Mumford stacks, \mathcal{X} of finite type and finite inertia over S , \mathcal{X}' of finite type and finite inertia over some Noetherian scheme S' . For every object \mathcal{Y} of \mathcal{C}/\mathcal{X} , fix a 2-base change $\mathcal{Y}' \rightarrow \mathcal{Y}$ of g . Then \mathcal{Y}' is of finite inertia over S' by 1.11. For every 1-cell $\mathcal{Z} \rightarrow \mathcal{Y}$ of \mathcal{C}/\mathcal{X} , fix a 2-Cartesian square of S -stacks obtained by 2-base change by g

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{g''} & \mathcal{Z} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g'} & \mathcal{Y} \end{array}$$

In this way, we have defined a functor $\mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}_{S'}$. Let F_1 be the composition $\mathcal{C}/\mathcal{X} \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$ and F_2 be the composition $\mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}_{S'} \xrightarrow{F_{S'}} \mathcal{D}$, $\epsilon_0: |F_1| \rightarrow |F_2|$ be given by $\epsilon_0(\mathcal{Y}) = g'^*$. Let $\epsilon_A: F_1|A_{\mathcal{X}} \Rightarrow F_2|A_{\mathcal{X}}$ be the pseudo-natural transformation with $|\epsilon_A| = \epsilon_0$ given by the inverse of the base change map (1.4.1), $\epsilon_B: F_1|B_{\mathcal{X}} \Rightarrow F_2|B_{\mathcal{X}}$ be pseudo-natural transformation with $|\epsilon_B| = \epsilon_0$ given by proper base change. It follows from 7.24 and 7.25 that (ϵ_A, ϵ_B) is a 1-cell of $\text{GD}'_{A,B_{\mathcal{X}}}(\mathcal{C}/\mathcal{X}, \mathcal{D})$. Thus it induces a pseudo-natural transformation $\epsilon: F_1 \Rightarrow F_2$ with $|\epsilon| = \epsilon_0$. For any 1-cell $f: \mathcal{Z} \rightarrow \mathcal{Y}$ of \mathcal{C}/\mathcal{X} , $\epsilon(f)$ is a natural equivalence

$$g'^* Rf! \Rightarrow Rf'_! g''^*.$$

(iii) *Projection formula isomorphism.* Let \mathcal{X} be a Deligne-Mumford S -stack of finite type and finite inertia, $N \in D(\mathcal{X}, \Lambda)$. Let F_1 be the composition $\mathcal{C}/\mathcal{X} \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$ as in (ii) and let $\epsilon_0: |F_1| \rightarrow |F_1|$ be given by $\epsilon_0(\mathcal{Y}) = N_{\mathcal{Y}} \otimes_{\Lambda}^L -$, where $N_{\mathcal{Y}}$ is the pull back of N to \mathcal{Y} . Let $\epsilon_A: F_1|A_{\mathcal{X}} \Rightarrow F_1|A_{\mathcal{X}}$ be the pseudo-natural transformation with $|\epsilon_A| = \epsilon_0$ given by the inverse of the projection formula (1.3.3), $\epsilon_B: F_1|B_{\mathcal{X}} \Rightarrow F_1|B_{\mathcal{X}}$ be pseudo-natural transformation with $|\epsilon_B| = \epsilon_0$ given by the projection formula 1.15 (ii). Let \mathcal{E} be the category whose objects are 1 and 2 and whose morphisms are $\mathbf{1}_1, \mathbf{1}_2$ and $s: 1 \rightarrow 2$. Consider the pseudo-functor $\mathcal{C}/\mathcal{X} \times \mathcal{E} \rightarrow \mathcal{D}$ given by

$$(\mathcal{Y}, i) \mapsto D(\mathcal{Y}, \Lambda), \quad (\mathbf{1}_{\mathcal{Y}}, s) \mapsto N_{\mathcal{Y}} \otimes_{\Lambda}^L -, \quad (f, \mathbf{1}_i) \mapsto f^* -,$$

where $i = 1, 2$. Then (1.3.3) and 1.15 (ii) are respectively the base change maps 7.20 and 7.18. Hence 7.24 and 7.25 imply that (ϵ_A, ϵ_B) is a 1-cell of $\text{GD}'_{A,B_{\mathcal{X}}}(\mathcal{C}/\mathcal{X}, \mathcal{D})$. Thus it induces a pseudo-natural transformation $\epsilon: F_1 \Rightarrow F_1$ with $|\epsilon| = \epsilon_0$. For any 1-cell $f: \mathcal{Z} \rightarrow \mathcal{Y}$ of \mathcal{C}/\mathcal{X} , $\epsilon(f)$ is a natural equivalence

$$N_{\mathcal{Y}} \otimes_{\Lambda}^L Rf! - \Rightarrow Rf!(N_{\mathcal{Z}} \otimes_{\Lambda}^L -).$$

We have obtained the following.

Theorem 1.18. (i) *(Base change) For any 2-Cartesian square of Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

with \mathcal{X} and \mathcal{Y} of finite type and finite inertia over S , \mathcal{Y}' of finite type and finite inertia over some Noetherian scheme S' , f separated of prime to m inertia, the base change map

$$g^* Rf! M \rightarrow Rf'_! h^* M$$

is an isomorphism for all $M \in D(\mathcal{X}, \Lambda)$.

(ii) (Projection formula) Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford stacks of finite type and finite inertia over S , $M \in D(\mathcal{X}, \Lambda)$, $N \in D(\mathcal{Y}, \Lambda)$. Assume that f has prime to m inertia. Then the projection formula map

$$N \otimes_{\Lambda}^L Rf_! M \rightarrow Rf_!(f^* N \otimes_{\Lambda}^L M)$$

is an isomorphism.

Corollary 1.19. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford stacks of finite type and finite inertia over S . Assume that f has prime to m inertia and the fibers of f have dimension $\leq d$. Then the cohomological amplitude of $Rf_!: D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$ is contained in $[0, 2d]$.

Proof. By 1.18 (i), we may assume that \mathcal{Y} is the spectrum of a field. If we decompose $f \simeq pj\pi$ as in 1.10 with j dominant, then $j_!$ is exact, π_* is exact and p_* has cohomological dimension $\leq 2d$ by 1.14, because the source of p has dimension $\leq d$. \square

Proposition 1.20. There exists a unique way to define, for every separated quasi-finite flat representable morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of Deligne-Mumford stacks of finite type and finite inertia over some Noetherian scheme T and every sheaf of Λ -modules \mathcal{F} on \mathcal{Y} , a trace map

$$\mathrm{Tr}_f(\mathcal{F}): Rf_! f^* \mathcal{F} \rightarrow \mathcal{F}$$

satisfying the following conditions:

(a) (Compatibility with base change) For every 2-Cartesian square of Deligne-Mumford stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

with \mathcal{X} and \mathcal{Y} of finite type and finite inertia over some Noetherian scheme T , \mathcal{X}' and \mathcal{Y}' of finite type and finite inertia over some Noetherian scheme T' , f separated quasi-finite flat representable, and every $\mathcal{F} \in \mathrm{Mod}(\mathcal{Y}, \Lambda)$, the following diagram commutes

$$\begin{array}{ccc} g^* Rf_! f^* \mathcal{F} & \xrightarrow{g^* \mathrm{Tr}_f(\mathcal{F})} & g^* \mathcal{F} \\ \downarrow c & \mathrm{Tr}_{f'}(g^* \mathcal{F}) \uparrow & \\ Rf'_! h^* f^* \mathcal{F} & \xrightarrow{\simeq} & Rf'_! f'^* g^* \mathcal{F} \end{array}$$

where c is the base change isomorphism.

(b) (Compatibility with composition) For every pair of composable separated quasi-finite flat representable morphisms

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$$

of Deligne-Mumford stacks of finite type and finite inertia over some Noetherian scheme T and every $\mathcal{F} \in \mathrm{Mod}(\mathcal{Z}, \Lambda)$, the following diagram commutes

$$\begin{array}{ccc} R(gf)_! (gf)^* \mathcal{F} & \xrightarrow{\mathrm{Tr}_{gf}(\mathcal{F})} & \mathcal{F} \\ \downarrow \simeq & \mathrm{Tr}_f(\mathcal{F}) \uparrow & \\ Rg_! Rf_! f^* g^* \mathcal{F} & \xrightarrow{Rg_! \mathrm{Tr}_f(g^* \mathcal{F})} & Rf_! f^* \mathcal{F} \end{array}$$

(c) (Normalization) If f is finite flat of constant degree d , the composition

$$\mathcal{F} \xrightarrow{a} f_* f^* \mathcal{F} \xrightarrow{\simeq} Rf_! f^* \mathcal{F} \xrightarrow{\mathrm{Tr}_f(\mathcal{F})} \mathcal{F},$$

where a is the adjunction, is multiplication by d .

Moreover, $\mathrm{Tr}_f(\mathcal{F})$ is functorial in \mathcal{F} .

Note that $Rf_!$ has cohomological dimension 0 and is the derived functor of $R^0 f_!$. Since the construction of Tr_f is local for the étale topology on \mathcal{Y} , (1.20) follows from [SGA 4, XVII 6.2.3].

In a different setting, Olsson [Ols08, 4.1] constructed trace maps for morphisms that are not necessarily representable.

1.21. Let $j: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated étale representable morphism of Deligne-Mumford S -stacks of finite type and finite inertia. Then the composition of natural transformations of functors $\mathrm{Mod}(\mathcal{X}, \Lambda) \rightarrow \mathrm{Mod}(\mathcal{Y}, \Lambda)$

$$Rj_! \xrightarrow{Rj_! a} Rj_! j^* j_! \xrightarrow{\mathrm{Tr}_j j_!} j_!,$$

where $a: \mathbf{1} \Rightarrow j^* j_!$ is the adjunction (1.5), is a natural equivalence. In fact, by base change, we are reduced to the trivial case where \mathcal{Y} is the spectrum of a separably closed field. More generally, we have a natural equivalence $Rj_! \simeq j_!$ of functors $D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$.

Proposition 1.22. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford S -stacks of finite type and finite inertia. Assume that f is of prime to m inertia. Then $Rf_!$ admits a right adjoint $Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$. In particular, for $K \in D(\mathcal{X}, \Lambda)$ and $L \in D(\mathcal{Y}, \Lambda)$, we have an isomorphism*

$$(1.22.1) \quad \mathrm{Hom}_{\mathcal{Y}}(Rf_! K, L) \rightarrow \mathrm{Hom}_{\mathcal{X}}(K, Rf^! L),$$

functorial in K and L .

If f is a closed immersion, $Rf^!$ is isomorphic to the functor defined in 1.6. If f is étale and representable, 1.21 induces a natural equivalence $f^* \simeq Rf^!$.

Proof. This is a formal consequence of the fact that $Rf_!$ commutes with small direct sums [KS06, 14.3.1 (ix)]. We may also repeat the construction of $Rf^!$ in [SGA 4, XVIII 3.1.4] as follows, at least when Λ is a Noetherian ring. Choose a decomposition $f \simeq pj\pi$ as in 1.10 and an integer d , upper bound of the dimensions of the fibers of f . For $\mathcal{F} \in \mathrm{Mod}(\mathcal{X}, \Lambda)$, define

$$f_!^\bullet \mathcal{F} = p_* \tau_{\leq 2d} \mathcal{C}_\ell^\bullet(j_! \pi_* \mathcal{F}),$$

where \mathcal{C}_ℓ^\bullet is the modified Godement resolution (2.6). Then $f_!^\bullet \mathcal{F}$ computes $Rf_! \mathcal{F}$. For every q , the functor $f_!^q: \mathrm{Mod}(\mathcal{X}, \Lambda) \rightarrow \mathrm{Mod}(\mathcal{Y}, \Lambda)$ is exact and commutes with small inductive limits, hence admits a right adjoint [SGA 4, XVIII 3.1.3]

$$f_q^!: \mathrm{Mod}(\mathcal{Y}, \Lambda) \rightarrow \mathrm{Mod}(\mathcal{X}, \Lambda)$$

that preserves injectives. The functor

$$\mathrm{tot} f_\bullet^!: C(\mathrm{Mod}(\mathcal{Y}, \Lambda)) \rightarrow C(\mathrm{Mod}(\mathcal{X}, \Lambda))$$

passes through a functor of homotopy categories, which has a right localization [KS06, 14.3.1 (vi)]

$$Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda).$$

For $K \in C(\mathrm{Mod}(\mathcal{X}, \Lambda))$ and $M \in C(\mathrm{Mod}(\mathcal{Y}, \Lambda))$, adjunction induces an isomorphism of complexes

$$(1.22.2) \quad \mathrm{Hom}_{\mathcal{Y}}^\bullet(\mathrm{tot} f_!^\bullet K, M) \rightarrow \mathrm{Hom}_{\mathcal{X}}^\bullet(K, \mathrm{tot} f_\bullet^! M).$$

It follows that $\mathrm{tot} f_\bullet^!$ preserves homotopically injective complexes. The isomorphism (1.22.1) is induced by (1.22.2), where M is a homotopically injective resolution of L . \square

Proposition 1.23. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be as in 1.22, K be in $D(\mathcal{X}, \Lambda)$, L and M be in $D(\mathcal{Y}, \Lambda)$. We have natural isomorphisms*

$$(1.23.1) \quad R\mathcal{H}om_{\mathcal{Y}}(Rf_! K, L) \rightarrow Rf_* R\mathcal{H}om_{\mathcal{X}}(K, Rf^! L),$$

$$(1.23.2) \quad R\mathcal{H}om_{\mathcal{X}}(f^* M, Rf^! L) \rightarrow Rf^! R\mathcal{H}om_{\mathcal{Y}}(M, L),$$

functorial in K , L and M .

Proof. These are induced by the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Y}}(M, R\mathcal{H}om_{\mathcal{Y}}(Rf_! K, L)) &\simeq \mathrm{Hom}_{\mathcal{Y}}(M \otimes_{\Lambda}^L Rf_! K, L) \\ &\xrightarrow{A^{-1}} \mathrm{Hom}_{\mathcal{Y}}(Rf_!(f^* M \otimes_{\Lambda}^L K), L) \simeq \mathrm{Hom}_{\mathcal{X}}(f^* M \otimes_{\Lambda}^L K, Rf^! L) \\ &\simeq \mathrm{Hom}_{\mathcal{X}}(f^* M, \mathcal{H}om_{\mathcal{X}}(K, Rf^! L)) \simeq \mathrm{Hom}_{\mathcal{Y}}(M, Rf_* R\mathcal{H}om_{\mathcal{X}}(K, Rf^! L)), \\ \mathrm{Hom}_{\mathcal{X}}(K, R\mathcal{H}om_{\mathcal{X}}(f^* M, Rf^! L)) &\simeq \mathrm{Hom}_{\mathcal{X}}(f^* M \otimes_{\Lambda}^L K, Rf^! L) \\ &\simeq \mathrm{Hom}_{\mathcal{Y}}(Rf_!(f^* M \otimes_{\Lambda}^L K), L) \xrightarrow{A} \mathrm{Hom}_{\mathcal{Y}}(M \otimes_{\Lambda}^L Rf_! K, L) \\ &\simeq \mathrm{Hom}_{\mathcal{Y}}(Rf_! K, R\mathcal{H}om_{\mathcal{Y}}(M, L)) \simeq \mathrm{Hom}_{\mathcal{X}}(K, Rf^! R\mathcal{H}om_{\mathcal{Y}}(M, L)), \end{aligned}$$

where A is induced by projection formula $M \otimes_{\Lambda}^L Rf_! K \rightarrow Rf_!(f^* \otimes_{\Lambda}^L K)$ (1.18 (ii)). \square

The base change isomorphism 1.17 (ii) induces the following by adjunction.

Proposition 1.24. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-Cartesian square of Deligne-Mumford S -stacks of finite type and finite inertia, g separated of prime to m inertia. Let $M \in D(\mathcal{X}, \Lambda)$. Then the map

$$Rf'_* Rf'^! M \rightarrow Rg^! Rf_* M$$

is an isomorphism.

1.25. The construction of $Rf_!$ in 1.17 works without assumption on the inertia of f if we restrict to D^+ . More precisely, let \mathcal{C}_1 be the 2-category of Deligne-Mumford S -stacks of finite type and finite inertia whose 1-cells are the separated morphisms, \mathcal{A} be the 2-faithful subcategory of \mathcal{C}_1 whose 1-cells are the open immersions, \mathcal{B}_1 be the 2-faithful subcategory of \mathcal{C}_1 whose 1-cells are the proper morphisms, \mathcal{D} be the 2-category of triangulated categories. Let $F_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{D}$ be the pseudo-functor given by 1.5:

$$\mathcal{X} \mapsto D^+(\mathcal{X}, \Lambda), \quad j \mapsto j_!, \quad \alpha \mapsto \alpha_!,$$

and let $F_{\mathcal{B}_1}: \mathcal{B}_1 \rightarrow \mathcal{D}$ be the pseudo-functor given by

$$\mathcal{X} \mapsto D^+(\mathcal{X}, \Lambda), \quad p \mapsto Rf_*, \quad \alpha \mapsto R\alpha_*.$$

We define G and ρ as before. Note that the proper base change map (1.17.1) is an isomorphism because we work over D^+ . This defines an object of $\mathrm{GD}'_{\mathcal{A}, \mathcal{B}_1}(\mathcal{C}_1, \mathcal{D})$ and let $F: \mathcal{C}_1 \rightarrow \mathcal{D}$ be a corresponding pseudo-functor. For any 1-cell $f: \mathcal{X} \rightarrow \mathcal{Y}$ of \mathcal{C}_1 , we define

$$Rf_{\dagger}: D^+(\mathcal{X}, \Lambda) \rightarrow D^+(\mathcal{Y}, \Lambda)$$

to be $F(f)$. If f is of prime to m to inertia, this is the same as $Rf_!$. In general, Rf_{\dagger} is not the correct definition of $Rf_!$, but in § 5 we will use it to give the correct definition of $Rf_!$ for coefficients of characteristic 0.

For any 1-cell $f: \mathcal{X} \rightarrow \mathcal{Y}$ of \mathcal{C}_1 , we have the support-forgetting natural transformation of functors $D^+(\mathcal{X}, \Lambda) \rightarrow D^+(\mathcal{Y}, \Lambda)$

$$Rf_{\dagger} \Rightarrow Rf_*,$$

which is a natural equivalence if f is proper. Base change isomorphisms are also constructed as before.

Proposition 1.26. *For any 2-Cartesian square of Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

with \mathcal{X} and \mathcal{Y} of finite type and finite inertia over S , \mathcal{Y}' of finite type and finite inertia over some Noetherian scheme S' , f separated, the base change map

$$g^* Rf_{\dagger} M \rightarrow Rf'_{\dagger} h^* M$$

is an isomorphism for all $M \in D^+(\mathcal{X}, \Lambda)$.

Proposition 1.27. *Assume that S has finite dimension. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prime to m inertia of Deligne-Mumford S -stacks of finite type.*

- (i) $f_*: \text{Mod}(\mathcal{X}, \Lambda) \rightarrow \text{Mod}(\mathcal{Y}, \Lambda)$ *has finite cohomological dimension.*
- (ii) *If f is a closed immersion, $Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$ has finite cohomological amplitude.*
- (iii) *Assume that Λ is annihilated by an integer m invertible on S . If f is separated of prime to m inertia, then $Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$ has finite cohomological amplitude.*

Proof. (i) Up to replacing \mathcal{Y} by an étale presentation and \mathcal{X} by the corresponding 2-pull-back, we may assume that \mathcal{Y} is a scheme. In particular, if f is an open immersion, then the result follows from the case of schemes, which is a recent result of Gabber [Gab]. This implies (ii) by the distinguished triangle (1.6.2). In the general case, we proceed by induction. Let $j: \mathcal{U} \rightarrow \mathcal{X}$ be a dominant open immersion with \mathcal{U} separated over S and let i be a complementary closed immersion. Applying Rf_* to (1.6.2) and induction hypothesis to fi , we are reduced to prove that $(fj)_*$ has finite cohomological dimension. In other words, we may assume \mathcal{X} separated over S . By the factorization 1.10, we are then reduced to two cases: (a) f is an open immersion; (b) f is proper. Case (a) is known and (b) follows from 1.14.

(ii) Already proved.

(iii) We easily reduce to two cases: (a) f is a smooth morphism of schemes; (b) f is a closed immersion. In case (a), $Rf^! \simeq f^*(d)[2d]$, where d is the relative dimension. Case (b) is an instance of (ii). \square

2 Operations on $D_c^b(\mathcal{X}, \Lambda)$

In 2.1 through 2.7, let S be a quasi-separated scheme and Λ be a Noetherian ring.

2.1. On a Deligne-Mumford S -stack \mathcal{X} , we say a sheaf $\mathcal{F} \in \text{Mod}(\mathcal{X}, \Lambda)$ is *constructible* if $\alpha^* \mathcal{F} \in \text{Mod}(X, \Lambda)$ is constructible for some (or, equivalently, for every) étale presentation $\alpha: X \rightarrow \mathcal{X}$ where X an S -scheme.

We say that a full subcategory of an abelian category is *thick* [KS06, 8.3.21 (iv)] if it is closed under kernels, cokernels and extensions. The full subcategory $\text{Mod}_c(\mathcal{X}, \Lambda)$ of $\text{Mod}(\mathcal{X}, \Lambda)$ consisting of constructible sheaves Λ -modules is thick. Let $D_c(\mathcal{X}, \Lambda)$ be the triangulated subcategory of $D(\mathcal{X}, \Lambda)$ consisting of complexes with constructible cohomology sheaves.

Proposition 2.2. *Let \mathcal{X} be a quasi-compact Deligne-Mumford S -stack. Denote by $\text{Etqc}(\mathcal{X})$ the full subcategory of $\text{Et}(\mathcal{X})$ consisting of Deligne-Mumford S -stacks representable, étale and quasi-compact over \mathcal{X} . Let \mathcal{F} be a sheaf of Λ -modules \mathcal{F} on \mathcal{X} . Then the following conditions are equivalent:*

- (a) \mathcal{F} is constructible;
- (b) For any epimorphism $\alpha: \mathcal{F}_I = \bigoplus_{i \in I} \Lambda_{\mathcal{U}_i, \mathcal{X}} \rightarrow \mathcal{G}$, where \mathcal{U}_i is an object of $\text{Etqc}(\mathcal{X})$ for every $i \in I$, \mathcal{G} is either \mathcal{F} or the kernel of a map $\Lambda_{\mathcal{V}, \mathcal{X}} \rightarrow \mathcal{F}$, \mathcal{V} is an object of $\text{Etqc}(\mathcal{X})$, there exists a finite subset $J \subset I$ such that $\mathcal{F}_J \hookrightarrow \mathcal{F}_I \xrightarrow{\alpha} \mathcal{G}$ is an epimorphism, where $\mathcal{F}_J = \bigoplus_{j \in J} \Lambda_{\mathcal{U}_j, \mathcal{X}}$;
- (c) \mathcal{F} is isomorphic to the cokernel of a map $\Lambda_{\mathcal{U}, \mathcal{X}} \rightarrow \Lambda_{\mathcal{V}, \mathcal{X}}$, where \mathcal{U} and \mathcal{V} are objects of $\text{Etqc}(\mathcal{X})$;

Here $\Lambda_{\mathcal{U}, \mathcal{X}}$ is the sheaf defined in 1.3.

Proof. Let $f: X \rightarrow \mathcal{X}$ be a quasi-compact étale presentation where X is a scheme.

(c) \implies (a). Since $f^* \Lambda_{\mathcal{U}, \mathcal{X}}$ is constructible, so is $\Lambda_{\mathcal{U}, \mathcal{X}}$.

(a) \implies (b). By the above, \mathcal{G} is constructible. Since $f^* \alpha: f^* \mathcal{F}_I \rightarrow f^* \mathcal{G}$ is an epimorphism, there exists a finite subset $J \subset I$ such that $f^* \mathcal{F}_J \rightarrow f^* \mathcal{G}$ is an epimorphism by the proof of [SGA 4, IX 2.7]. Then $\mathcal{F}_J \rightarrow \mathcal{G}$ is an epimorphism.

(b) \implies (c). Note that $(\Lambda_{\mathcal{U}, \mathcal{X}})_{\mathcal{U} \in \text{Etqc}(\mathcal{X})}$ is a system of generators of $\text{Mod}(\mathcal{X}, \Lambda)$. Thus there exists an epimorphism $\mathcal{F}_I \rightarrow \mathcal{F}$. By (b), we may assume that I is finite. Then $\mathcal{F}_I \simeq \Lambda_{\mathcal{V}, \mathcal{X}}$, where $\mathcal{V} = \coprod_{i \in J} \mathcal{U}_i \in \text{Etqc}(\mathcal{X})$. Applying the above argument to $\mathcal{G} = \text{Ker}(\Lambda_{\mathcal{V}, \mathcal{X}} \rightarrow \mathcal{F})$, we get an epimorphism $\Lambda_{\mathcal{U}, \mathcal{X}} \rightarrow \mathcal{G}$, $\mathcal{U} \in \text{Etqc}(\mathcal{X})$. \square

Corollary 2.3. *Let \mathcal{X} be a quasi-compact Deligne-Mumford S -stack, \mathcal{F} be a constructible Λ -module on \mathcal{X} . The functor $\text{Hom}_{\mathcal{X}}(\mathcal{F}, -): \text{Mod}(\mathcal{X}, \Lambda) \rightarrow \Lambda$ commutes with small inductive limits.*

This follows from 2.2 and [SGA 4, VI 1.23].

Corollary 2.4. *Let \mathcal{X} be a quasi-compact Deligne-Mumford S -stack. Any Λ -module \mathcal{F} on \mathcal{X} is a filtrant inductive limit of constructible Λ -modules on \mathcal{X} .*

Proof. Fix an epimorphism $\mathcal{F}_I = \bigoplus_{i \in I} \Lambda_{\mathcal{U}_i, \mathcal{X}} \rightarrow \mathcal{F}$, where \mathcal{U}_i is an object of $\text{Etqc}(\mathcal{X})$, $i \in I$. For every $i \in I$, fix a surjection $\mathcal{F}_{J_i} \rightarrow \text{Ker}(\mathcal{F}_i \rightarrow \mathcal{F})$. Then \mathcal{F} is the inductive limit of $\text{Coker}(\mathcal{F}_B \rightarrow \mathcal{F}_A)$, where (A, B) runs over pairs of finite subsets $A \subset I$, $B \subset \coprod_{i \in I} J_i$. The pairs are ordered by inclusion. \square

Corollary 2.5. *Let \mathcal{X} be a quasi-compact Deligne-Mumford S -stack. The functor*

$$\varinjlim: \text{Ind Mod}_c(\mathcal{X}, \Lambda) \rightarrow \text{Mod}(\mathcal{X}, \Lambda)$$

is an equivalence of categories.

This follows from 2.3 and 2.4.

2.6. As in [SGA 4, XVIII 3.1.2], 2.5 allows us to construct the modified Godement resolution. Let \mathcal{X} be a quasi-compact Deligne-Mumford S -stack. For any sheaf \mathcal{F} of Λ -modules on \mathcal{X} , denote the Godement resolution of \mathcal{F} by $\mathcal{C}^\bullet(\mathcal{F})$. Define the modified Godement resolution to be

$$\mathcal{C}_\ell^\bullet(\mathcal{F}) = \varinjlim_{i \in I} \mathcal{C}^\bullet(\mathcal{F}_i),$$

where $(\mathcal{F}_i)_{i \in I}$ is a system of constructible Λ -modules such that $\mathcal{F} \simeq \varinjlim_{i \in I} \mathcal{F}_i$. This is a flasque resolution of \mathcal{F} , independent up to isomorphisms of the choice of $(\mathcal{F}_i)_{i \in I}$. For all q , the functor $\mathcal{C}_\ell^q(\mathcal{F})$ is exact and commutes with small inductive limits and étale localization.

Proposition 2.7. *Let \mathcal{X} be a Noetherian Deligne-Mumford S -stack. A sheaf of Λ -modules \mathcal{F} on \mathcal{X} is constructible if and only if it is Noetherian.*

In particular, constructible sheaves on \mathcal{X} are closed by subobjects and quotients.

Proof. Let $f: X \rightarrow \mathcal{X}$ be a quasi-compact étale presentation where X is an S -scheme. If \mathcal{F} is constructible, then $f^* \mathcal{F}$ is constructible, hence Noetherian [SGA 4, 2.9 (ii)]. It follows that \mathcal{F} is Noetherian. If \mathcal{F} is Noetherian, then the sheaf \mathcal{G} in 2.2 (b) is Noetherian by the above, thus \mathcal{F} is constructible by 2.2 (b) \implies (a). \square

In the rest of this section, let S be a regular scheme of dimension ≤ 1 . See 5.7 for a remark on this assumption. In 2.8 and 2.9, let Λ be a Noetherian ring annihilated by an integer m invertible on S .

Proposition 2.8. *Let \mathcal{X} be a Deligne-Mumford S -stack of finite type. The functor*

$$- \otimes -: D(\mathcal{X}, \Lambda) \times D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$$

sends $D_c^- \times D_c^-$ to D_c^- and

$$R\mathcal{H}om_{\mathcal{X}}: D(\mathcal{X}, \Lambda)^{\text{op}} \times D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$$

sends $(D_c^-)^{\text{op}} \times D_c^+$ to D_c^+ .

Proof. Take an étale presentation $X \rightarrow \mathcal{X}$ with X a separated finite type S -scheme. It suffices to verify the assertions for X , which are classical [SGA 4 $\frac{1}{2}$, Th. fin. 1.6]. \square

Proposition 2.9. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford S -stacks of finite type.*

(i) f^ sends $D_c(\mathcal{Y}, \Lambda)$ to $D_c(\mathcal{X}, \Lambda)$, Rf_* sends $D_c^+(\mathcal{X}, \Lambda)$ to $D_c^+(\mathcal{Y}, \Lambda)$. If f is of prime to m inertia, then Rf_* sends $D_c(\mathcal{X}, \Lambda)$ to $D_c(\mathcal{Y}, \Lambda)$.*

(ii) If f is a representable and étale, then $f_!$ (1.5) sends $D_c(\mathcal{X}, \Lambda)$ to $D_c(\mathcal{Y}, \Lambda)$.

(iii) If f is a closed immersion, then $Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$ (1.6) sends $D_c(\mathcal{Y}, \Lambda)$ to $D_c(\mathcal{X}, \Lambda)$.

(iv) Assume f is separated, \mathcal{X} and \mathcal{Y} are of finite inertia. Then Rf_{\dagger} sends $D_c^+(\mathcal{X}, \Lambda)$ to $D_c^+(\mathcal{Y}, \Lambda)$. If, moreover, f has prime to m inertia, then $Rf_!$ sends $D_c(\mathcal{X}, \Lambda)$ to $D_c(\mathcal{Y}, \Lambda)$ and $Rf^!$ sends $D_c(\mathcal{Y}, \Lambda)$ to $D_c(\mathcal{X}, \Lambda)$.

Proof. (i) The result for f^* is trivial. For Rf_* , up to replacing \mathcal{Y} by an étale presentation and \mathcal{X} by the corresponding pull-back, we may assume that \mathcal{Y} is a scheme. In particular, if f is an open immersion, then the result follows from the case of schemes, which is classical [SGA 4 $\frac{1}{2}$, Th. fin. 1.3]. This implies (iii) by the distinguished triangle (1.6.2). In the general case, we reduce by Chow's lemma and cohomological descent to the known case where \mathcal{X} is also a scheme.

(iii) This has been proved in the proof of (i).

(iv) For $Rf^!$, we reduce to (iii) and the case of a smooth morphism of schemes. In this case, $Rf^! \simeq f^*(d)[2d]$, where d is the relative dimension of f . For Rf_{\dagger} and $Rf_!$, by construction we are reduced to two cases: (a) f is an open immersion; (b) f is proper. Case (a) is clear while (b) is a special case of (i).

(ii) Replacing \mathcal{Y} by an open covering, we may assume \mathcal{Y} separated over S . If $\{\mathcal{U}_1, \mathcal{U}_2\}$ is an open covering of \mathcal{X} , we have the Mayer-Vietoris distinguished triangle

$$j_{0!}j_0^*K \rightarrow j_{1!}j_1^*K \oplus j_{2!}j_2^*K \rightarrow K \rightarrow,$$

where $j_i: \mathcal{U}_i \rightarrow \mathcal{X}$ is the open immersion, $i = 0, 1, 2$, $\mathcal{U}_0 = \mathcal{U}_1 \cap \mathcal{U}_2$, for any $K \in D(\mathcal{X}, \Lambda)$. Thus we may assume \mathcal{X} separated over S . It then suffices to apply (iv) and the comparison 1.21. \square

2.10. In the rest of this section let Λ be a Gorenstein ring of dimension 0 annihilated by an integer m invertible on S . The case of main interest is $\Lambda = \mathcal{O}/\mathfrak{m}^{n+1}$, where \mathcal{O} a complete discrete valuation ring of mixed characteristic and \mathfrak{m} is the maximal ideal of \mathcal{O} .

Let $\Omega_S \in D_c^b(S, \Lambda)$ be the object such that $\Omega_S[T] = \Lambda(d_T)[2d_T]$ for every connected component T of S , where $d_T = \dim T$. For any finite type and separated morphism $a: X \rightarrow S$ of schemes, let $\Omega_X = Ra^!\Omega_S \in D_c^b(X, \Lambda)$. Define a triangulated functor

$$D_X: D(X, \Lambda)^{\text{op}} \rightarrow D(X, \Lambda)$$

by $D_X M = R\mathcal{H}om_{\Lambda}(M, \Omega_X)$. Let $d_X = \max_{s \in S}(\dim X_s + d_s)$, where d_s is the dimension of the closure of s in S .

Lemma 2.11. (i) The functor D_X induces a functor $(D_c^-)^{\text{op}} \rightarrow D_c^+$, whose cohomological amplitude is contained in $[-2d_X, 0]$. In particular, $\Omega_X \in D_c^{[-2d_X, 0]}(X, \Lambda)$ and D_X sends $(D_c^b)^{\text{op}}$ to D_c^b .

(ii) Ω_X is a dualizing complex for X [SGA 5, I 1.7].

In particular, $\text{Ext}^i(\Omega_X, \Omega_X) = H^i(X, D_X \Omega_X) \simeq H^i(X, \Lambda) = 0$ for all $i < 0$.

Proof. (i) It follows from 2.8 that D_X sends $(D_c^-)^{\text{op}}$ to D_c^+ . The bound can be obtained similarly to the proof of [SGA 4 $\frac{1}{2}$, Fin. 1.6]. In fact, for $\mathcal{F} \in \text{Mod}_c(X, \mathcal{F})$, to show $D_X \mathcal{F}$ belongs to $D^{[-2d_X, 0]}$, we may assume $\mathcal{F} = j_! \mathcal{G}$, where $j: Y \rightarrow X$ is an immersion, Y is regular and connected and \mathcal{G} is a lisse sheaf on Y . Then $D_X j_! \mathcal{G} \simeq Rj_* D_Y \mathcal{G}$ by (1.3.2) and

$$(D_Y \mathcal{G})_{\bar{y}} = R\mathcal{H}om_{\Lambda}(\mathcal{G}_{\bar{y}}, \Lambda(d_Y)[2d_Y])$$

for any geometric point $\bar{y} \rightarrow Y$. We then apply the fact that Rj_* has cohomological dimension $\leq 2d_Y$.

(ii) It remains to show that for $M \in D_c^b(X, \Lambda)$, the map $M \rightarrow D_X D_X M$ is an isomorphism. The proof is identical to the proof of [SGA 4 $\frac{1}{2}$, Fin. 4.3]. \square

Let \mathcal{X} be a finite type Deligne-Mumford S -stack. We apply the ‘‘BBD gluing lemma’’ [BBD82, 3.2.4] to define a dualizing complex $\Omega_{\mathcal{X}}$. For every étale morphism $\alpha: X \rightarrow \mathcal{X}$ with X a separated finite type S -scheme, we associate Ω_X . For any morphism $f: X \rightarrow Y$ between such morphisms, there is a canonical isomorphism $f^* \Omega_Y \xrightarrow{\sim} \Omega_X$. Since, for all X , Ω_X belongs to $D^{[-2d_X, 0]}(X, \Lambda)$, where $d_X = \max_{s \in S} (\dim \mathcal{X}_s + d_s)$, there exists a unique $\Omega_{\mathcal{X}} \in D^{[-2d_{\mathcal{X}}, 0]}(\mathcal{X}, \Lambda)$ such that, for every étale morphism $\alpha: X \rightarrow \mathcal{X}$ with X a separated finite type S -scheme, we have $\alpha^* \Omega_{\mathcal{X}} \simeq \Omega_X$. It follows that $\Omega_{\mathcal{X}}$ belongs to $D_c^{[-2d_{\mathcal{X}}, 0]}$. Define a triangulated functor

$$D_{\mathcal{X}}: D(\mathcal{X}, \Lambda)^{\text{op}} \rightarrow D(\mathcal{X}, \Lambda)$$

by $D_{\mathcal{X}} M = R\mathcal{H}om_{\Lambda}(M, \Omega_{\mathcal{X}})$.

Proposition 2.12. (i) The functor $D_{\mathcal{X}}$ induces a functor $(D_c^-)^{\text{op}} \rightarrow D_c^+$ whose cohomological amplitude is contained in $[-2d_{\mathcal{X}}, 0]$. In particular $D_{\mathcal{X}}$ sends $(D_c^b)^{\text{op}}$ to D_c^b .

(ii) For $M \in D_c^b(\mathcal{X}, \Lambda)$, the natural map $M \rightarrow D_{\mathcal{X}} D_{\mathcal{X}} M$ is an isomorphism.

Proof. Take an étale presentation $X \rightarrow \mathcal{X}$ with X a separated finite type S -scheme. It suffices to verify the proposition for X , which is classical (2.11). \square

2.13. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of prime to m inertia of Deligne-Mumford S -stacks of finite type. Assume either f is a closed immersion, or \mathcal{X} and \mathcal{Y} are of finite inertia. The functor $Rf^!: D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$ as defined in 1.6 and 1.22 preserves D_c^b . The isomorphisms (1.6.3) and (1.23.2) give an isomorphism $D_{\mathcal{X}} f^* \xrightarrow{\sim} Rf^! D_{\mathcal{Y}}$ of functors $D(\mathcal{Y}, \Lambda) \rightarrow D(\mathcal{X}, \Lambda)$. Using biduality (2.12 (ii)), we obtain an isomorphism of functors $D_c^b(\mathcal{Y}, \Lambda) \rightarrow D_c^b(\mathcal{X}, \Lambda)$:

$$Rf^! \simeq Rf^! D_{\mathcal{Y}} D_{\mathcal{Y}} \simeq D_{\mathcal{X}} f^* D_{\mathcal{X}}.$$

2.14. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Deligne-Mumford S -stacks. Thanks to 2.12, we can define a triangulated functor

$$Rf^!: D_c^b(\mathcal{Y}, \Lambda) \rightarrow D_c^b(\mathcal{X}, \Lambda), \quad N \mapsto D_{\mathcal{X}} f^* D_{\mathcal{Y}} N.$$

By 2.13, this definition is compatible with 1.6 and 1.22. If $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ are two such morphisms, then, by biduality, we have an isomorphism of functors $D_c^b(\mathcal{Z}, \Lambda) \rightarrow D_c^b(\mathcal{X}, \Lambda)$:

$$R(gf)^! = D_{\mathcal{X}}(gf)^* D_{\mathcal{Z}} \simeq D_{\mathcal{X}} f^* g^* D_{\mathcal{Z}} \simeq D_{\mathcal{X}} f^* D_{\mathcal{Y}} D_{\mathcal{Y}} g^* D_{\mathcal{Z}} = Rf^! Rg^!.$$

If f is smooth, it follows from the construction of $\Omega_{\mathcal{X}}$ and $\Omega_{\mathcal{Y}}$ that $\Omega_{\mathcal{X}} \simeq f^* \Omega_{\mathcal{Y}}(d)[2d]$, where d is the relative dimension of f . It follows that (1.8.1) induces an isomorphism $f^*(D_{\mathcal{Y}} N)(d)[2d] \simeq D_{\mathcal{X}} f^* N$ for all $N \in D(\mathcal{X}, \Lambda)$. Thus

$$Rf^! = D_{\mathcal{X}} f^* D_{\mathcal{Y}} \simeq f^*(d)[2d].$$

2.15. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prime to m inertia of Deligne-Mumford S -stacks of finite type. Assume either (a) f is representable and étale, or (b) f is separated and \mathcal{X} and \mathcal{Y} are of finite inertia. The functor $Rf_!: D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$ as defined in 1.5 and 1.17 preserves D_c^b . The isomorphisms (1.3.2) and (1.23.1) give an isomorphism $D_{\mathcal{Y}}Rf_! \xrightarrow{\sim} Rf_*D_{\mathcal{X}}$ of functors $D(\mathcal{X}, \Lambda) \rightarrow D(\mathcal{Y}, \Lambda)$. Using biduality (2.12 (ii)), we obtain an isomorphism of functors $D_c^b(\mathcal{X}, \Lambda) \rightarrow D_c^b(\mathcal{Y}, \Lambda)$:

$$Rf_! \simeq D_{\mathcal{Y}}D_{\mathcal{Y}}Rf_! \simeq D_{\mathcal{Y}}Rf_*D_{\mathcal{X}}.$$

2.16. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prime to m inertia between Deligne-Mumford S -stacks of finite type. Thanks to 2.12, we can define a triangulated functor

$$Rf_!: D_c^b(\mathcal{X}, \Lambda) \rightarrow D_c^b(\mathcal{Y}, \Lambda), \quad M \mapsto D_{\mathcal{Y}}Rf_*D_{\mathcal{X}}M.$$

By 2.15, this definition is compatible with both 1.5 and 1.17. If $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ are two such morphisms, then, by biduality, we have an isomorphism of functors $D_c^b(\mathcal{X}, \Lambda) \rightarrow D_c^b(\mathcal{Z}, \Lambda)$:

$$R(gf)_! = D_{\mathcal{Z}}R(gf)_*D_{\mathcal{X}} \simeq D_{\mathcal{Z}}Rg_*Rf_*D_{\mathcal{X}} \simeq D_{\mathcal{Z}}Rg_*D_{\mathcal{Y}}D_{\mathcal{Y}}Rf_*D_{\mathcal{X}} \simeq Rg_!Rf_!.$$

3 Construction of $D_c(\mathcal{X}, \mathcal{O})$

In this section we fix a complete discrete valuation ring \mathcal{O} . Let $\mathfrak{m} = \lambda\mathcal{O}$ be the maximal ideal of \mathcal{O} and $\Lambda_n = \mathcal{O}/\mathfrak{m}^{n+1}$.

3.1. Let \mathcal{X} be a topos. Consider the topos $\mathcal{X}^{\mathbb{N}}$ of projective systems $(M_{\bullet} = (M_n)_{n \in \mathbb{N}})$ of sheaves on \mathcal{X} and the ring $\Lambda_{\bullet} = (\Lambda_n)_{n \in \mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ whose transition maps $\Lambda_{n+1} \rightarrow \Lambda_n$ are induced by the identity map on \mathcal{O} . Let $(\pi_*, \pi^*) : (\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow (\mathcal{X}, \mathcal{O})$ be the morphism of ringed toposes defined by $\pi_*(F_{\bullet}) = \varprojlim_n F_n$ and $\pi^*G = (\Lambda_n \otimes_{\mathcal{O}} G)_{n \in \mathbb{N}}$.

For all n , let $e_n : \mathcal{X} \rightarrow \mathcal{X}_{\text{ét}}^{\mathbb{N}}$ be the morphism of toposes defined by $e_n^{-1}(G_{\bullet}) = G_n$, $(e_n^*F)_q = F$ for $q \geq n$, and $(e_n^*F)_q = \{*\}$ for $q < n$. It induces a flat morphism of ringed toposes $(\mathcal{X}, \Lambda_n) \rightarrow (\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$. Note that $e_{n*} : \text{Mod}(\mathcal{X}, \Lambda_n) \rightarrow \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$ is exact and $e_n^{-1}e_{n*} = \mathbf{1}$. For $M \in D(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$, we denote $e_n^{-1}M$ by M_n . The functor $e_n^{-1} : \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow \text{Mod}(\mathcal{X}, \Lambda_n)$ admits a left adjoint $e_{n!}$ given by $(e_n!F)_q = \Lambda_q \otimes_{\Lambda_n} F$ for $q \leq n$ and $(e_n!F)_m = 0$ for $q > n$. The composition $\pi e_n : (\mathcal{X}, \Lambda_n) \rightarrow (\mathcal{X}, \mathcal{O})$ is the ring change morphism and we often omit $(\pi e_n)_*$ to simplify notations.

We say that $M \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$ is *essentially zero* if for every $n \geq 0$, there exists $r \geq 0$ such that the transition map $M_{n+r} \rightarrow M_n$ is zero. We say that M is *AR-null* [SGA 5, V 2.2.1] if, moreover, r can be chosen independently of n . The full subcategory

$$\mathcal{N} \subset \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$$

consisting of AR-null modules is a thick subcategory closed by subobjects and quotients. We say that $M \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$ is *preadic* (“ \mathfrak{m} -adic” in the terminology of [SGA 5, V 3.1.1]) if, for all n , the natural map $\Lambda_n \otimes_{\Lambda_{n+1}} M_{n+1} \rightarrow M_n$ is an isomorphism; $M \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$ is *AR-preadic* (“AR- \mathfrak{m} -adic” in the terminology of [SGA 5, V 3.2.2]) if there exists a preadic module $N \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$ and a homomorphism $N \rightarrow M$ with AR-null kernel and cokernel.

Lemma 3.2. *Let $L, M \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$, L preadic. Then the localization map*

$$\text{Hom}_{\text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})}(L, M) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})/\mathcal{N}}(L, M)$$

is an isomorphism.

It follows that $M \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$ is AR-preadic if and only if there exists a preadic module $L \in \text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})$, isomorphic to M in $\text{Mod}(\mathcal{X}_{\text{ét}}^{\mathbb{N}}, \Lambda_{\bullet})/\mathcal{N}$.

Proof. Let \mathcal{N}' be the full subcategory of $\text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})$ consisting of AR-null systems in that category [SGA 5, V 2.2.1]. The functor

$$\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})/\mathcal{N} \rightarrow \text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})/\mathcal{N}'$$

induced by the inclusion functor is faithful. Thus it suffices to show that the localization map

$$\text{Hom}_{\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})}(L, M) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})/\mathcal{N}'}(L, M)$$

is an isomorphism. By [SGA 5, V 2.4.4 (iv)], we have

$$\text{Hom}_{\text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})/\mathcal{N}'}(L, M) \simeq \varinjlim_{r \in \mathbb{N}} \text{Hom}_{\text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})}(L(r), M),$$

where $L(r)$ is the translation of L given by $L(r)_n = L_{n+r}$. Since L is preadic and M is a Λ_{\bullet} -module, the map

$$\text{Hom}_{\text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})}(L, M) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{X}^{\mathbb{N}}, \pi^{-1}\mathcal{O})}(L(r), M)$$

induced by the transition map $L(r) \rightarrow L$ is an isomorphism for every r . \square

3.3. We say that $M \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is *essentially zero* (resp. *AR-null*) if $\mathcal{H}^i M$ is essentially zero (resp. AR-null) for all i . Let $D_{\mathcal{N}}$ be the full subcategory of $D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ consisting of AR-null complexes. It is a thick triangulated subcategory [Ric89, 1.3].

Lemma 3.4. (a) For $M \in \text{Mod}(\mathcal{X}, \mathcal{O})$, $\pi^* M$ is preadic, $\mathcal{H}^{-1} L\pi^* M$ is essentially zero, and $\mathcal{H}^{-q} L\pi^* M = 0$ for $q > 1$.

(b) For $N \in D(\mathcal{X}, \Lambda_n)$, the natural map $L\pi^* N \rightarrow e_{n*} N$ has AR-null cone. In particular, for $N \in \text{Mod}(\mathcal{X}, \Lambda_n)$, $\mathcal{H}^{-1} L\pi^* N$ is AR-null.

For $M \in D(\mathcal{X}, \mathcal{O})$, the distinguished triangle

$$L\pi^* \tau^{\leq q} M \rightarrow L\pi^* M \rightarrow L\pi^* \tau^{\geq q+1} M \rightarrow$$

induces the short exact sequence

$$(3.4.1) \quad 0 \rightarrow \pi^* \mathcal{H}^q M \rightarrow \mathcal{H}^q L\pi^* M \rightarrow \mathcal{H}^{-1} L\pi^* \mathcal{H}^{q+1} M \rightarrow 0$$

By (a), $\pi^* \mathcal{H}^q M$ is preadic and $\mathcal{H}^{-1} L\pi^* \mathcal{H}^{q+1} M$ is essentially zero.

Proof. (a) The first assertion follows from the isomorphism $\Lambda_n \otimes_{\Lambda_{n+1}} \Lambda_{n+1} \otimes_{\Lambda_{\bullet}} M \simeq \Lambda_n \otimes_{\Lambda_{\bullet}} M$. The last assertion holds because $e_n^{-1} \mathcal{H}^{-q} L\pi^* M \simeq \mathcal{T}or_q^{\mathcal{O}}(\Lambda_n, M) = 0$ for $q > 1$. Consider the short exact sequence of $\pi^{-1}\mathcal{O}$ -modules $0 \rightarrow F \rightarrow \pi^{-1}\mathcal{O} \rightarrow \Lambda_{\bullet} \rightarrow 0$, where $F = (\mathcal{O})_{m \in \mathbb{N}}$ has transition maps $F_{m+1} \rightarrow F_m$ given by $\mathcal{O} \xrightarrow{\times \lambda} \mathcal{O}$, $F_n \rightarrow (\pi^{-1}\mathcal{O})_n$ is given by $\mathcal{O} \xrightarrow{\times \lambda^{n+1}} \mathcal{O}$. This sequence gives a $\pi^{-1}\mathcal{O}$ -flat resolution of Λ_{\bullet} . So $\mathcal{H}^{-1} L\pi^* M$ is a sub- $\pi^{-1}\mathcal{O}$ -module of $F \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1} M$, hence is essentially zero.

(b) For $N \in \text{Mod}(\mathcal{X}, \Lambda_n)$, $\mathcal{H}^{-1} L\pi^* N$ is a sub- $\pi^{-1}\mathcal{O}$ -module of $F \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1} N$, hence is AR-null. For the first assertion, by (a) we may assume $N \in \text{Mod}(\mathcal{X}, \Lambda_n)$. In this case $\pi^* N \rightarrow e_{n*} N$ is an epimorphism with AR-null kernel. \square

The following is a variant of [Eke90, 1.4].

Lemma 3.5. Let $M \in D^-(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be AR-null, $N \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. Assume either $M \in D^b$ or $N \in D^-$. Then $M \otimes_{\Lambda_{\bullet}}^L N$ is AR-null.

Proof. We may assume $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. It then suffices to take a quasi-isomorphism $N' \rightarrow N$, where N' belongs to the smallest triangulated subcategory of $K(\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}))$ stable under small direct sums and containing complexes bounded above with flat components. \square

Lemma 3.6. *Suppose that \mathcal{X} has enough points. Let $M \in D(\mathcal{X}, \Lambda_n)$ and $N \in D(\mathcal{X}, \mathcal{O})$. Then the projection formula map*

$$e_{n*} M \otimes_{\Lambda_\bullet}^L L\pi^* N \rightarrow e_{n*} (M \otimes_{\Lambda_n}^L e_n^{-1} L\pi^* N)$$

is an isomorphism.

Proof. We may assume that \mathcal{X} is the punctual topos. Since $e_{n*}: \text{Mod}(\mathcal{X}, \Lambda_n) \rightarrow \text{Mod}(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ commutes with small direct limits, we may repeat the arguments in the proof of 1.15 (ii). \square

Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of toposes. It induces a flat morphism of ringed toposes

$$(f_*^\mathbb{N}, f^{\mathbb{N}*}): (\mathcal{X}^\mathbb{N}, \Lambda_\bullet) \rightarrow (\mathcal{Y}^\mathbb{N}, \Lambda_\bullet).$$

The base change map $Le_{n!} f^* \Rightarrow f^{\mathbb{N}*} Le_{n!}$ is a natural equivalence of functors $D(\mathcal{Y}, \Lambda_n) \rightarrow D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$. By adjunction (7.21), we have the following.

Lemma 3.7. *For all $M \in D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$, the base change map*

$$(3.7.1) \quad e_n^{-1} Rf_*^\mathbb{N} M \rightarrow Rf_* e_n^{-1} M$$

is an isomorphism.

The case $M \in D^+$ is a particular case of [SGA 4, V^{bis} 1.3.12]. It follows also from the fact that the full subcategory $I_\mathcal{X}$ of $\text{Mod}(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ consisting of modules of the form $\prod_n e_{n*} I_{(n)}$, where $I_{(n)}$ is an injective Λ_n -module, is $f_*^\mathbb{N}$ -injective. For $I \in I_\mathcal{X}$, I is injective and $e_n^{-1} I$ is injective for all n .

The lemma implies that $Rf_*^\mathbb{N}$ preserves essentially zero complexes in D^+ and AR-null complexes in D^+ .

The base change maps induce a natural transformation of functors $D(\mathcal{X}, \Lambda_n) \rightarrow D(\mathcal{Y}^\mathbb{N}, \Lambda_\bullet)$

$$(3.7.2) \quad Le_{n!} Rf_* \Rightarrow Rf_*^\mathbb{N} Le_{n!}.$$

Lemma 3.8. (a) *If $M \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ is essentially zero, we have $R\pi_* M = 0$.*

(b) *For $N \in D^+(\mathcal{X}, \Lambda_n)$, the adjunction map $N \rightarrow R\pi_* L\pi^* N$ is an isomorphism.*

Part (a) is a particular case of [Eke90, 1.1].

Proof. (a) Note that $R^q \pi_* M$ is the sheaf associated to the presheaf $(U \mapsto H^q(U^\mathbb{N}, M))$, where U runs over objects of \mathcal{X} . Let $a: U \rightarrow \text{pt}$ be the morphism of toposes from U to the punctual topos. Since $Ra_*^\mathbb{N} M$ is essentially zero, $R\Gamma(U^\mathbb{N}, M) \simeq R\varprojlim R a_*^\mathbb{N} M = 0$.

(b) By (a) and 3.4 (b), it suffices to show that the natural map $N \rightarrow R\pi_* e_{n*} N$ is an isomorphism, which is trivial. \square

3.9. We define a functor

$$(3.9.1) \quad D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet) \rightarrow D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet), \quad M \mapsto \hat{M} = L\pi^* R\pi_* M.$$

Following Ekedahl [Eke90, 2.1 (iii)] we say that a complex $M \in D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ is *normalized* if the adjunction map $\hat{M} \rightarrow M$ is an isomorphism in $D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$.

The following criterion is a variant of [Eke90, 2.2 (ii)] and plays an essential role in what follows.

Lemma 3.10. *Let $M \in D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$. Consider the following conditions:*

(a) *M is normalized;*

(b) *$M \simeq L\pi^* N$ for some $N \in D(\mathcal{X}, \mathcal{O})$;*

(c) *For all n , the natural map*

$$(3.10.1) \quad \Lambda_n \otimes_{\Lambda_{n+1}}^L M_{n+1} \rightarrow M_n$$

is an isomorphism.

Then (a) \implies (b) \implies (c). Moreover, if $M \in D^+$, then they are equivalent.

In particular, for $M \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, \hat{M} is normalized.

Proof. (a) \implies (b) By definition, $M \simeq L\pi^* R\pi_* M$.

(b) \implies (c) We have $M_n \simeq \Lambda_n \otimes_{\mathcal{O}}^L N$ and (3.10.1) is induced by the isomorphism

$$\Lambda_n \otimes_{\Lambda_{n+1}}^L \Lambda_{n+1} \otimes_{\mathcal{O}}^L N \xrightarrow{\sim} \Lambda_n \otimes_{\mathcal{O}}^L N.$$

(c) \implies (a) assuming $M \in D^+$. To prove that M is normalized, it suffices to show that the map $e_n^{-1} \hat{M} \rightarrow M_n$ is an isomorphism for all n . This map is the composition

$$(3.10.2) \quad e_n^{-1} \hat{M} \xrightarrow{\sim} \Lambda_n \otimes_{\mathcal{O}}^L R\pi_* M \xrightarrow{\alpha} R\pi_*(L\pi^* \Lambda_n \otimes_{\Lambda_{\bullet}}^L M) \xrightarrow{\beta} R\pi_* e_{n*} M_n \simeq M_n,$$

where α is the projection formula map and β is induced by the composition of

$$(3.10.3) \quad L\pi^* \Lambda_n \otimes_{\Lambda_{\bullet}}^L M \rightarrow \pi^* \Lambda_n \otimes_{\Lambda_{\bullet}}^L M$$

and the adjunction map

$$(3.10.4) \quad \pi^* \Lambda_n \otimes_{\Lambda_{\bullet}}^L M \rightarrow e_{n*} e_n^{-1} (\pi^* \Lambda_n \otimes_{\Lambda_{\bullet}}^L M) \simeq e_{n*} M_n.$$

Taking the resolution $F \rightarrow \Lambda_n$, where F is the complex $\mathcal{O} \xrightarrow{\times \lambda^{n+1}} \mathcal{O}$ concentrated in degrees -1 and 0 , we see that α is an isomorphism. By 3.4 (b) and 3.5, (3.10.3) has AR-null cone. By (c),

$$\Lambda_n \otimes_{\Lambda_m}^L M_m \rightarrow M_n$$

is an isomorphism for all $m \geq n$. It follows that (3.10.4) has AR-null cone. Note that $L\pi^* \Lambda_n \otimes_{\Lambda_{\bullet}}^L M$ and $e_{n*} M_n$ are both in D^+ . Hence β is an isomorphism by 3.8 (a). \square

Lemma 3.11. *Let $M \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be normalized. Then the map*

$$L\pi^* \tau^{\geq a} R\pi_* M \rightarrow \widehat{\tau^{\geq a} M}$$

is an isomorphism.

Proof. By 3.4 (a), the map

$$L\pi^* \tau^{\geq a} R\pi_* M \rightarrow \tau^{\geq a} L\pi^* R\pi_* M \simeq \tau^{\geq a} M$$

has essentially zero cone. By 3.10 and 3.8 (a), the maps

$$L\pi^* \tau^{\geq a} R\pi_* M \rightarrow (L\pi^* \tau^{\geq a} R\pi_* M)^{\wedge} \rightarrow \widehat{\tau^{\geq a} M}$$

are isomorphisms. \square

Proposition 3.12. *Let $M, N \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be normalized. Then $M \otimes_{\Lambda_{\bullet}}^L N$ is normalized.*

Proof. By assumption, $M \simeq L\pi^* M'$, $N \simeq L\pi^* N'$ for some $M', N' \in D^+(\mathcal{X}, \mathcal{O})$. Thus

$$M \otimes_{\Lambda_{\bullet}}^L N \simeq L\pi^* M' \otimes_{\Lambda_{\bullet}}^L L\pi^* N' \simeq L\pi^* (M' \otimes_{\mathcal{O}}^L N').$$

\square

Proposition 3.13. *Suppose \mathcal{X} has enough points. Let $N \in D^{\geq a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be normalized. Then*

- (i) $-\otimes_{\Lambda_{\bullet}} N$ has cohomological amplitude $\geq a - 1$.
- (ii) For all $M \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ AR-null, $M \otimes_{\Lambda_{\bullet}} N$ is AR-null.

Proof. By assumption, $N \simeq L\pi^*N'$ for some $N' \in D^{\geq a}(\mathcal{X}, \mathcal{O})$. Then $M \otimes_{\Lambda_\bullet}^L N \simeq M \otimes_{\pi^{-1}\mathcal{O}}^L \pi^{-1}N'$. Then (i) follows from the fact that $-\otimes_{\pi^{-1}\mathcal{O}}^L -$ has cohomological amplitude ≥ -1 . For (ii), we are then reduced to the case $M \in D^b$, which is covered by 3.5. \square

Proposition 3.14. *Let $M \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$, $N \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$. Suppose that either*

(a) *M is normalized; or*

(b) *the cohomology sheaves of M are preadic, $M \in D^-$ and $N \in D^+$.*

Then the natural map

$$e_n^{-1} R\mathcal{H}om_{\Lambda_\bullet}(M, N) \rightarrow R\mathcal{H}om_{\Lambda_n}(M_n, N_n)$$

is an isomorphism.

This is a variant of [LO08b, 3.1.2].

Proof. Let $j_n: \mathcal{X}^{\leq n} \rightarrow \mathcal{X}^{\mathbb{N}}$ be the morphism of toposes defined by $j_n^{-1}M = (M_m)_{m \leq n}$, $(j_n^*N)_m = N_m$ for $m \leq n$, $(j_n^*N)_m = N_n$ for $m \geq n$. Let $e'_n: \mathcal{X} \rightarrow \mathcal{X}^{\leq n}$ be the morphism of toposes defined by $e'_n{}^{-1}N = N_n$, $(e'_{n*}F)_m = \{*\}$ for $m < n$, $(e'_{n*}F)_n = F$. Then $e_n = j_n e'_n$. Let $\pi_n: (\mathcal{X}^{\leq n}, \Lambda_{\leq n}) \rightarrow (\mathcal{X}, \Lambda_n)$ be the morphism of ringed toposes given by $\pi_n^* = e'_n{}^{-1}$, $(\pi_n^*F)_m = \Lambda_m \otimes_{\Lambda_n} F$.

In case (b), we may assume $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ is preadic, then $\pi_n^*M_n \xrightarrow{\sim} j_n^{-1}M$. As observed in the remark following 3.7, there is a quasi-isomorphism $N \rightarrow I$ such that I^k and $e_n^{-1}I^k$ are injective for all k and n . Hence it suffices to show that for any $N \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$, $e_n^{-1}\mathcal{H}om_{\Lambda_\bullet}(M, N) \rightarrow \mathcal{H}om_{\Lambda_n}(M_n, N_n)$ is an isomorphism. This is clear because the map is the composition

$$\begin{aligned} e_n^{-1}\mathcal{H}om_{\Lambda_\bullet}(M, N) &= e'_n{}^{-1}j_n^{-1}\mathcal{H}om_{\Lambda_\bullet}(M, N) \xrightarrow{\sim} e'_n{}^{-1}\mathcal{H}om_{j_n^{-1}\Lambda_\bullet}(j_n^{-1}M, j_n^{-1}N) \\ &\xrightarrow{\sim} \pi_n^*\mathcal{H}om_{\Lambda_{\leq n}}(\pi_n^*M_n, j_n^{-1}N) \simeq \mathcal{H}om_{\Lambda_n}(M_n, \pi_n^*j_n^{-1}N) \simeq \mathcal{H}om_{\Lambda_n}(M_n, N_n). \end{aligned}$$

In case (a), $L\pi_n^*M_n \xrightarrow{\sim} j_n^{-1}M$ by 3.10. Hence

$$\begin{aligned} e_n^{-1}R\mathcal{H}om_{\Lambda_\bullet}(M, N) &= e'_n{}^{-1}j_n^{-1}R\mathcal{H}om_{\Lambda_\bullet}(M, N) \xrightarrow{\sim} e'_n{}^{-1}R\mathcal{H}om_{j_n^{-1}\Lambda_\bullet}(j_n^{-1}M, j_n^{-1}N) \\ &\xrightarrow{\sim} \pi_n^*R\mathcal{H}om_{\Lambda_{\leq n}}(L\pi_n^*M_n, j_n^{-1}N) \simeq R\mathcal{H}om_{\Lambda_n}(M_n, \pi_n^*j_n^{-1}N) \simeq R\mathcal{H}om_{\Lambda_n}(M_n, N_n). \end{aligned}$$

\square

Corollary 3.15. *Let $M \in D^-(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ and $N \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ be normalized. Then*

$$R\mathcal{H}om_{\Lambda_\bullet}(M, N) \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$$

is normalized.

Proof. By 3.10 and 3.14, we need to show that

$$(3.15.1) \quad \Lambda_n \otimes_{\Lambda_{n+1}}^L R\mathcal{H}om_{\Lambda_{n+1}}(M_{n+1}, N_{n+1}) \rightarrow R\mathcal{H}om_{\Lambda_n}(M_n, N_n)$$

is an isomorphism. By assumption, $\Lambda_n \otimes_{\mathcal{O}}^L M' \xrightarrow{\sim} M_n$, $\Lambda_n \otimes_{\mathcal{O}}^L N' \xrightarrow{\sim} N_n$, where $M' = R\pi_*M$ and $N' = R\pi_*N \in D(\mathcal{X}, \mathcal{O})$. Thus (3.15.1) is isomorphic to

$$\Lambda_n \otimes_{\Lambda_{n+1}}^L R\mathcal{H}om_{\Lambda_{n+1}}(\Lambda_{n+1} \otimes_{\mathcal{O}}^L M', \Lambda_{n+1} \otimes_{\mathcal{O}}^L N') \rightarrow R\mathcal{H}om_{\Lambda_n}(\Lambda_n \otimes_{\mathcal{O}}^L M', \Lambda_n \otimes_{\mathcal{O}}^L N').$$

Therefore, it suffices to show that

$$\Lambda_n \otimes_{\mathcal{O}}^L R\mathcal{H}om_{\mathcal{O}}(M', N') \rightarrow R\mathcal{H}om_{\Lambda_n}(\Lambda_n \otimes_{\mathcal{O}}^L M', \Lambda_n \otimes_{\mathcal{O}}^L N')$$

is an isomorphism. This map is the composition of the map

$$(3.15.2) \quad \Lambda_n \otimes_{\mathcal{O}}^L R\mathcal{H}om_{\mathcal{O}}(M', N') \rightarrow R\mathcal{H}om_{\mathcal{O}}(M', \Lambda_n \otimes_{\mathcal{O}}^L N')$$

and the adjunction isomorphism

$$R\mathcal{H}om_{\mathcal{O}}(M', \Lambda_n \otimes_{\mathcal{O}}^L N') \xrightarrow{\sim} R\mathcal{H}om_{\Lambda_n}(\Lambda_n \otimes_{\mathcal{O}}^L M', \Lambda_n \otimes_{\mathcal{O}}^L N').$$

Taking the resolution $F \rightarrow \Lambda_n$, where F is the complex $\mathcal{O} \xrightarrow{\times \lambda^{n+1}} \mathcal{O}$ concentrated in degrees -1 and 0 , we conclude that (3.15.2) is an isomorphism. \square

Proposition 3.16. *Let $N \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be AR-null and $M \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. Assume that one of the following conditions holds:*

- (a) $M \in D^-$ is normalized;
- (a') M is normalized and $N \in D^b$;
- (b) $M \in D^-$ is of preadic cohomology sheaves.

Then $R\mathcal{H}om_{\Lambda_{\bullet}}(M, N)$ is AR-null. Moreover, if (a) or (b) holds, then

$$R\mathcal{H}om_{\Lambda_{\bullet}}(M, N) = 0.$$

Proof. We may assume $N \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. In case (b), we may assume $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is preadic. In all cases, by 3.14,

$$e_n^{-1} R\mathcal{H}om_{\Lambda_{\bullet}}(M, N) \simeq R\mathcal{H}om_{\Lambda_n}(M_n, N_n),$$

hence $e_n^{-1} \mathcal{E}xt_{\Lambda_{\bullet}}^q(M, N) \simeq \mathcal{E}xt_{\Lambda_n}^q(M_n, N_n)$. The transition maps of $\mathcal{E}xt_{\Lambda_{\bullet}}^q(M, N)$ are induced by the canonical maps

$$R\mathcal{H}om_{\Lambda_m}(M_m, N_m) \rightarrow R\mathcal{H}om_{\Lambda_m}(M_m, N_n) \xrightarrow{\alpha} R\mathcal{H}om_{\Lambda_n}(M_n, N_n)$$

for $m \geq n$. In cases (a) and (a'), α is the composition

$$R\mathcal{H}om_{\Lambda_m}(M_m, N_n) \xrightarrow{\sim} R\mathcal{H}om_{\Lambda_n}(\Lambda_n \otimes_{\Lambda_m}^L M_m, N_n) \xrightarrow{\sim} R\mathcal{H}om_{\Lambda_n}(M_n, N_n).$$

In case (b), α is the composition

$$\mathcal{H}om_{\Lambda_m}^{\bullet}(M_m, N') \xrightarrow{\sim} \mathcal{H}om_{\Lambda_n}^{\bullet}(\Lambda_n \otimes_{\Lambda_m} M_m, N') \xrightarrow{\sim} \mathcal{H}om_{\Lambda_n}^{\bullet}(M_n, N'),$$

where N' is an injective resolution of N_n . In all cases, since N is AR-null, $\mathcal{E}xt_{\Lambda_{\bullet}}^q(M, N)$ is AR-null for all q , hence $R\mathcal{H}om_{\Lambda_{\bullet}}(M, N)$ is AR-null. Therefore, if $M \in D^-$, then

$$R\mathcal{H}om_{\Lambda_{\bullet}}(M, N) \simeq R\Gamma(\mathcal{X}, R\pi_* R\mathcal{H}om_{\Lambda_{\bullet}}(M, N)) = 0$$

by 3.8 (a). \square

Corollary 3.17. *Let $M \in D^b(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, $N \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, M either normalized or of preadic cohomology sheaves. Let $D_{\mathcal{N}}^+ = D_{\mathcal{N}} \cap D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, where $D_{\mathcal{N}}$ is as in 3.3. Then the localization map*

$$\text{Hom}_{D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})}(M, N) \rightarrow \text{Hom}_{D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})/D_{\mathcal{N}}^+}(M, N)$$

is an isomorphism.

Proof. We have

$$\mathrm{Hom}_{D^+(\mathcal{X}, \Lambda_\bullet)/D_{\mathcal{N}}^+}(M, N) \simeq \varinjlim_{s: N \rightarrow N'} \mathrm{Hom}_{D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)}(M, N'),$$

where s runs over maps in $D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ with cone in $D_{\mathcal{N}}^+$. For every such s , it follows from 3.16 that the map

$$\mathrm{Hom}_{D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)}(M, N) \rightarrow \mathrm{Hom}_{D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)}(M, N')$$

induced by s is an isomorphism. \square

Proposition 3.18. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of toposes. The functors*

$$f^{\mathbb{N}*}: D^+(\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet) \rightarrow D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet), \quad Rf_*^{\mathbb{N}}: D(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet) \rightarrow D(\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet)$$

preserve normalized complexes.

Proof. The assertion for f^* follows trivially from 3.10. For Rf_* , consider the square of ringed toposes

$$\begin{array}{ccc} (\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet) & \xrightarrow{\pi_{\mathcal{X}}} & (\mathcal{X}, \mathcal{O}) \\ \downarrow f^{\mathbb{N}} & & \downarrow f \\ (\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet) & \xrightarrow{\pi_{\mathcal{Y}}} & (\mathcal{Y}, \mathcal{O}) \end{array}$$

By 3.10, it suffices to show that, for all $M \in D(\mathcal{X}, \mathcal{O})$, the base change map

$$L\pi_{\mathcal{Y}}^* Rf_* M \rightarrow Rf_*^{\mathbb{N}} L\pi_{\mathcal{X}}^* M$$

is an isomorphism. By 3.7, it suffices to show that the projection formula map of $D(X, \Lambda_n)$

$$(3.18.1) \quad \Lambda_n \otimes_{\mathcal{O}}^L Rf_* M \rightarrow Rf_*(\Lambda_n \otimes_{\mathcal{O}}^L M)$$

is an isomorphism. It then suffices to take the resolution $F \rightarrow \Lambda_n$, where F is the complex $\mathcal{O} \xrightarrow{\times \lambda^{n+1}} \mathcal{O}$ concentrated in degrees -1 and 0 . \square

3.19. Let U be an object of \mathcal{X} and $\mathcal{U} = \mathcal{X}/U$. Then $\mathcal{U}^{\mathbb{N}}$ can be identified with $\mathcal{X}^{\mathbb{N}}/U^{\mathbb{N}}$. Consider the morphism of toposes $j: \mathcal{U} \rightarrow \mathcal{X}$. The functor $j_!^{\mathbb{N}}: \mathrm{Mod}(\mathcal{U}^{\mathbb{N}}, \Lambda_\bullet) \rightarrow \mathrm{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ is a left adjoint of $j^{\mathbb{N}*}$ and is exact. It induces a triangulated functor $j_!^{\mathbb{N}}: D(\mathcal{U}^{\mathbb{N}}, \Lambda_\bullet) \rightarrow D(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$. The base change map

$$(3.19.1) \quad j_! e_n^{-1} \Rightarrow e_n^{-1} j_!^{\mathbb{N}}$$

is a natural equivalence of functors $D(\mathcal{U}^{\mathbb{N}}, \Lambda_\bullet) \rightarrow D(\mathcal{X}, \Lambda_n)$. In particular, $j_!^{\mathbb{N}}$ preserves AR-null complexes. Moreover, we have a natural equivalence of functors $D(\mathcal{U}, \Lambda_n) \rightarrow D(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$

$$(3.19.2) \quad j_!^{\mathbb{N}} L e_n! \Rightarrow L e_n! j_!.$$

It follows from 3.10 and projection formula (1.3.3) that $j_!^{\mathbb{N}}$ preserves normalized complexes in D^+ .

Proposition 3.20. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $\mathrm{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ with L and N AR-preadic. Then M is AR-preadic.*

This is a variant of [SGA 5, V 5.2.4]. In *loc. cit.*, one assumes moreover that L_n is Artinian for all n . This condition is seldom satisfied. Indeed, if there exists a nonzero Artinian Λ_n -module for some n on a scheme X of finite type over a field, then X is necessarily an Artinian scheme.

Proof. As in *loc. cit.*, we use the reductions in the proof of [SGA 5, V 3.2.4 (ii)] as follows. Let M' and N' be the universal image systems of M and N , respectively. Since L and N satisfy the Mittag-Leffler-Artin-Rees condition [SGA 5, V 3.2.3], M satisfies this condition too [SGA 5, V 2.1.2 (ii)]. So the monomorphisms $M' \rightarrow M$ and $N' \rightarrow N$ have AR-null cokernels. Let $L' = \text{Ker}(M' \rightarrow N')$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \end{array}$$

we see that $L' \rightarrow L$ is a monomorphism with AR-null cokernel. Therefore, we may assume that M and N are strict.

For $r \geq 0$, consider the functor $T_r: \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ defined by $(T_r F)_n = \Lambda_n \otimes_{\Lambda_{n+r}} F_{n+r}$. For F strict, the epimorphism $T_r F \rightarrow F$ has AR-null kernel. Since N is AR-preadic, there exists an $r \geq 0$ such that $T_r N$ is preadic [SGA 5, V 3.2.3]. Let $L'' = \text{Ker}(T_r M \rightarrow T_r N)$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L'' & \longrightarrow & T_r M & \longrightarrow & T_r N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \end{array}$$

we see that $L'' \rightarrow L$ has AR-null kernel and cokernel. Therefore, we may assume that M is strict and N is preadic.

In this case, L is strict [SGA 5, V 3.1.3 (ii)]. Choose $r \geq 0$ such that $T_r L$ is preadic. Then we have an exact sequence

$$0 \rightarrow T \rightarrow T_r L \rightarrow M \rightarrow N \rightarrow 0,$$

where $T = \text{Ker}(T_r L \rightarrow L)$ is AR-null. We are therefore reduced to the following lemma. \square

Lemma 3.21. *Let $0 \rightarrow T \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in $\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ with T AR-null, L and N preadic. Then M is AR-preadic.*

Proof. We decompose the exact sequence into two short exact sequences

$$(3.21.1) \quad 0 \rightarrow T \rightarrow L \rightarrow Q \rightarrow 0$$

and

$$(3.21.2) \quad 0 \rightarrow Q \rightarrow M \rightarrow N \rightarrow 0.$$

Using (3.21.1) and 3.16 (b), we obtain an isomorphism $\text{Ext}_{\Lambda_{\bullet}}^1(N, L) \xrightarrow{\sim} \text{Ext}_{\Lambda_{\bullet}}^1(N, Q)$. Taking a representative of the preimage of the class of (3.21.2), we get the 9-diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & T & \xlongequal{\quad} & T & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since L and N are preadic, M' is also preadic [SGA 5, V 3.1.3]. \square

3.22. We say that $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is *adic* (“Noetherian \mathfrak{m} -adic” in the terminology of [SGA 5, V 5.1.1]) if M is preadic and, for all n or, equivalently, for some n , M_n is Noetherian; $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is *AR-adic* if there exists an adic module $N \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ and a homomorphism $N \rightarrow M$ with AR-null kernel and cokernel. By 3.2, M is AR-adic if and only if there exists an adic module $N \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, isomorphic to M in $\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})/\mathcal{N}$. Note that we do not assume M_n to be Noetherian, hence our notion of AR-adic modules is more general than the notion of “Noetherian AR- \mathfrak{m} -adic projective systems” in [SGA 5, V 5.1.3]. Let $\text{Mod}_a(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ (resp. $\text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$) be the full subcategory of $\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ consisting of adic (resp. AR-adic) modules.

Proposition 3.23. *The category $\text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is a thick subcategory (2.1) of $\text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, that is, a full subcategory closed by kernel, cokernel and extension.*

Proof. Using the same methods as in [SGA 5, V 5.2.1], one shows that $\text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is stable by kernel and cokernel. Using the same reductions as in the proof of 3.20, one deduces from 3.21 that this subcategory is also stable by extension. \square

Define $\text{Mod}_c(\mathcal{X}, \mathcal{O})$ to be the quotient

$$\text{Mod}_c(\mathcal{X}, \mathcal{O}) = \text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})/\mathcal{N}.$$

By 3.2, the composition

$$\text{Mod}_a(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \xrightarrow{\iota} \text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \xrightarrow{\psi} \text{Mod}_c(\mathcal{X}, \mathcal{O})$$

of the inclusion functor ι and the localization functor ψ is an equivalence of categories. Let

$$\phi: \text{Mod}_c(\mathcal{X}, \mathcal{O}) \rightarrow \text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$$

be the composition of ι and a quasi-inverse of $\psi\iota$. By 3.2, ϕ is a left adjoint of ψ . Note that ϕ is not left exact unless \mathcal{X} is empty. By [SGA 5, V 5.2.3], $\text{Mod}_c(\mathcal{X}, \mathcal{O})$ is a Noetherian category.

Denote the category of Noetherian Λ_n -modules on \mathcal{X} by $\text{Mod}_c(\mathcal{X}, \Lambda_n)$. For $M \in \text{Mod}_c(\mathcal{X}, \mathcal{O})$, $N \in \text{Mod}_c(\mathcal{X}, \Lambda_n)$, we have

$$\text{Hom}_{\text{Mod}_c(\mathcal{X}, \Lambda_n)}(e_n^{-1}\phi M, N) \simeq \text{Hom}_{\text{Mod}_c(\mathcal{X}, \mathcal{O})}(M, e_n N).$$

Moreover, $e_n^{-1}\phi e_n \simeq e_n^{-1}\pi^* \simeq \mathbf{1}$. Thus $e_n^*: \text{Mod}_c(\mathcal{X}, \Lambda_n) \rightarrow \text{Mod}_c(\mathcal{X}, \mathcal{O})$ is fully faithful. Define $\Lambda_n \otimes_{\mathcal{O}} M = e_n^{-1}\phi M$. The family of functors

$$(\Lambda_n \otimes_{\mathcal{O}} -: \text{Mod}_c(\mathcal{X}, \mathcal{O}) \rightarrow \text{Mod}_c(\mathcal{X}, \Lambda_n))_{n \in \mathbb{N}}$$

is conservative.

We say $M \in \text{Mod}_a(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is *locally constant* if M_n is locally constant for all n . We say $M \in \text{Mod}_c(\mathcal{X}, \mathcal{O})$ is *lisse* if it is in the essential image of locally constant adic modules. For any n , this is equivalent to $\Lambda_n \otimes_{\mathcal{O}} M$ being locally constant.

3.24. We say that $M \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is *AR-adic* if $\mathcal{H}^i M$ is AR-adic for all i . Let $D_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be the full subcategory of $D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ consisting of AR-adic complexes, which is a thick triangulated subcategory [Ric89, 1.3]. Define $D_c^*(\mathcal{X}, \mathcal{O})$ to be the quotient

$$D_c^*(\mathcal{X}, \mathcal{O}) = D_{AR-a}^*(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})/D_{\mathcal{N}}^*,$$

where $*$ $\in \{\emptyset, +, b\}$, $D_{AR-a}^* = D_{AR-a} \cap D^*$, $D_{\mathcal{N}}^* = D_{\mathcal{N}} \cap D^*$, $D_{\mathcal{N}}$ as in 3.3. The inclusion functor induces a faithful functor $D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_c(\mathcal{X}, \mathcal{O})$ and a fully faithful functor $D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O})$ [KS06, 10.2.6]. The functor $\text{Mod}_c(\mathcal{X}, \mathcal{O}) \rightarrow D_c^b(\mathcal{X}, \mathcal{O})$ induced by the inclusion functor $\text{Mod}_{AR-a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D_{AR-a}^b(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is fully faithful by 3.2 and 3.17.

By 3.8 (a), the restriction of the functor (3.9.1) $M \mapsto \hat{M}$ to D^+ factors to give a functor $D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, called the *normalization functor*, that we still denote by $M \mapsto \hat{M}$.

In the rest of this section, assume that \mathcal{X} has enough points. We say that $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is *AR-torsion-free* if $\text{Ker}(M \xrightarrow{\times \lambda} M)$ is AR-null.

Lemma 3.25. *Let $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be preadic and AR-torsion-free. Then M is flat.*

Proof. We may assume that \mathcal{X} is the punctual topos. Then $M' = \varprojlim_n M_n$ is a torsion-free, hence flat \mathcal{O} -module and $M_n \simeq \Lambda_n \otimes_{\Lambda} M'$. \square

Proposition 3.26. *Let $M \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be AR-adic. Then the cone of $\hat{M} \rightarrow M$ is AR-null and $\mathcal{H}^q \hat{M}_n \in \text{Mod}(\mathcal{X}, \Lambda_n)$ is Noetherian for all q and n . In particular, \hat{M} is AR-adic. If, moreover, $M \in D^{[a,b]}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, then \hat{M} belongs to $D^{[a-1,b]}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$.*

This is similar to [LO08b, 3.0.14].

Proof. We may assume $M \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ AR-adic. From the fact that $\text{Mod}_c(\mathcal{X}, \mathcal{O})$ is Noetherian, one deduces that there exists $n \geq 0$ such that $M' = \text{Im}(M \xrightarrow{\times \lambda^n} M)$ is AR-torsion-free. Since M sits in the short exact sequence $0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$, where $M'' = \text{Ker}(M \xrightarrow{\times \lambda^n} M)$, we are reduced to the following cases

- (a) M is AR-torsion-free.
- (b) M is annihilated by λ^{n+1} for some $n \geq 0$.

In both cases, let $F \rightarrow M$ be a homomorphism of AR-null kernel and cokernel with F adic. Then the induced map $\hat{F} \rightarrow \hat{M}$ is an isomorphism. Therefore, to see that $\hat{M} \rightarrow M$ has AR-null cone, that $\mathcal{H}^q \hat{M}_n$ is Noetherian for all q and n , and that \hat{M} belongs to $D^{[-1,0]}$, we may assume moreover that M is adic.

In case (a), M is flat by 3.25, hence normalized by 3.10, that is $\hat{M} \xrightarrow{\sim} M$. In case (b), $M \simeq \pi^* N$, where $N = M_n \in \text{Mod}(\mathcal{X}, \Lambda_n)$ is Noetherian. By 3.4 (b), the cone C of $L\pi^* N \rightarrow \pi^* N$ is AR-null. Hence, by 3.8, the composition $N \rightarrow R\pi_* L\pi^* N \rightarrow R\pi_* \pi^* N$, where the first map is the adjunction map, is an isomorphism. Applying $L\pi^*$ on both sides, we obtain $L\pi^* N \xrightarrow{\sim} \hat{M}$. Note that $L\pi^* N$ belongs to $D^{[-1,0]}$ and, by the proof of 3.4, $e_n^{-1} \mathcal{H}^q L\pi^* N$ is a subquotient of N , hence is Noetherian, for all q and n . Hence \hat{M} belongs to $D^{[-1,0]}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ and $\mathcal{H}^q \hat{M}_n \in \text{Mod}(\mathcal{X}, \Lambda_n)$ is Noetherian for all q and n . The cone of $\hat{M} \rightarrow M$ is isomorphic to C , which is AR-null. \square

Corollary 3.27. *Let $M \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be normalized. Then, for any n , M is AR-adic if and only if $\mathcal{H}^q M_n$ is Noetherian for all q and n .*

Proof. If M is AR-adic, then $\mathcal{H}^q M_n \simeq \mathcal{H}^q \hat{M}_n$ is Noetherian by 3.26. Conversely, if $\mathcal{H}^q M_n$ is Noetherian for all q , we first show that $\widehat{\mathcal{H}^q M}$ belongs to $D^{[-1,0]}$ and is AR-adic for all q . The short exact sequence (3.4.1) for $R\pi_* M$ induces a short exact sequence

$$0 \rightarrow M' \rightarrow \mathcal{H}^q M \rightarrow M'' \rightarrow 0,$$

where M' is preadic and M'' is essentially zero. Since M'_n is Noetherian, M' is adic. By 3.8 (a), $\widehat{\mathcal{H}^q M} \simeq \widehat{M'}$. The latter belongs to $D^{[-1,0]}$ and is AR-adic by 3.26. The distinguished triangle

$$\widehat{\tau_{\leq q} M} \rightarrow M \rightarrow \widehat{\tau_{\geq q+1} M} \rightarrow$$

induces a short exact sequence

$$0 \rightarrow \mathcal{H}^0 \widehat{\mathcal{H}^q M} \rightarrow \mathcal{H}^q M \rightarrow \mathcal{H}^{-1} \widehat{\mathcal{H}^{q+1} M} \rightarrow 0.$$

Thus $\mathcal{H}^q M$ is AR-adic. \square

3.28. Let D_{norm}^+ be the full subcategory of $D_{AR-a}^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ consisting of normalized complexes and let $D_{\text{norm}}^b = D_{\text{norm}}^+ \cap D^b$. By 3.26, the normalization functor

$$D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_{\text{norm}}^+, \quad M \mapsto \hat{M}$$

is a quasi-section of the composition

$$D_{\text{norm}}^+ \rightarrow D_{AR-a}^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O})$$

of the inclusion and localization functors. In particular $D_c^+(\mathcal{X}, \mathcal{O})$ is a \mathfrak{U} -category, where \mathfrak{U} is the fixed universe. Thus, by 3.17, the composition

$$D_{\text{norm}}^b \xrightarrow{\iota} D_{AR-a}^b(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \xrightarrow{\psi} D_c^b(\mathcal{X}, \mathcal{O})$$

is an equivalence of categories, and the normalization functor is a quasi-inverse. Moreover, the composition of the localization functor and ι is a left adjoint of ψ .

Let $D_c^{+, \leq a}(\mathcal{X}, \mathcal{O})$ (resp. $D_c^{\geq a}(\mathcal{X}, \mathcal{O})$) be the essential image of $D_{AR-a}^{+, \leq a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ (resp. $D_{AR-a}^{\geq a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$) in $D_c^+(\mathcal{X}, \mathcal{O})$. Then $M \in D_c^+(\mathcal{X}, \mathcal{O})$ belongs to $D_c^{+, \leq a}(\mathcal{X}, \mathcal{O})$ if and only if $\hat{M} \in D^{\leq a}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. For $M \in D_c^{+, \leq a}(\mathcal{X}, \mathcal{O})$ and $N \in D_c^{\geq a+1}(\mathcal{X}, \mathcal{O})$,

$$\text{Hom}_{D_c^+(\mathcal{X}, \mathcal{O})}(M, N) = 0.$$

In fact, we may replace M by \hat{M} and N by $N' \in D^{\geq a+1}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. It then suffices to apply 3.17. Thus $(D_c^{+, \leq 0}, D_c^{\geq 0})$ is a t-structure. The cohomological amplitude of the normalization functor $M \mapsto \hat{M}$ is contained in $[-1, 0]$ and is not 0 unless \mathcal{X} is empty.

For $M \in D_c^b(\mathcal{X}, \mathcal{O})$ and $N \in D_c^+(\mathcal{X}, \mathcal{O})$, define $R\text{Hom}_{\mathcal{O}}(M, N) = R\text{Hom}_{\Lambda_{\bullet}}(\hat{M}, N)$. By 3.16, this gives a functor

$$R\text{Hom}_{\mathcal{O}}(-, -): D_c^b(\mathcal{X}, \mathcal{O})^{\text{op}} \times D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D^+(\mathcal{O}),$$

where $D^+(\mathcal{O})$ is the derived category of \mathcal{O} -modules. By 3.17, for $M \in D_c^b(\mathcal{X}, \mathcal{O})$ and $N \in D_c^+(\mathcal{X}, \mathcal{O})$, we have

$$H^0 R\text{Hom}_{\mathcal{O}}(M, N) \simeq \text{Hom}_{D_c^+(\mathcal{X}, \mathcal{O})}(M, N).$$

Let $D_c^+(\mathcal{X}, \Lambda_n)$ be the full subcategory of $D^+(\mathcal{X}, \Lambda_n)$ consisting of complexes with Noetherian cohomology sheaves. For $M \in D_c^b(\mathcal{X}, \mathcal{O})$, $N \in D_c^+(\mathcal{X}, \Lambda_n)$, we have

$$\text{Hom}_{D_c^+(\mathcal{X}, \Lambda_n)}(e_n^{-1} \hat{M}, N) \simeq \text{Hom}_{D_c^+(\mathcal{X}, \mathcal{O})}(M, e_{n*} N).$$

For $N \in D_c^b(\mathcal{X}, \Lambda_n)$, $e_n^{-1} \widehat{e_{n*} N} \simeq e_n^{-1} L\pi^* N \simeq N$. Thus $e_{n*}: D_c^b(\mathcal{X}, \Lambda_n) \rightarrow D_c^b(\mathcal{X}, \mathcal{O})$ is fully faithful. For $N \in D_c^+(\mathcal{X}, \mathcal{O})$, define $\Lambda_n \otimes_{\mathcal{O}}^L N = e_n^{-1} \hat{N}$. The functor

$$\Lambda_n \otimes_{\mathcal{O}}^L -: D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \Lambda_n)$$

is conservative.

4 Operations on $D_c^b(\mathcal{X}, \mathcal{O})$

In this section, let S be a regular scheme of dimension ≤ 1 , \mathcal{O} be a complete discrete valuation ring of residue characteristic ℓ invertible on S . See 5.7 for a remark on the assumption on S . As in § 3, let $\Lambda_n = \mathcal{O}/\mathfrak{m}^n$, where \mathfrak{m} is the maximal ideal of \mathcal{O} . Let \mathcal{X} be a finite type Deligne-Mumford S -stack. We apply the formalism of § 3 to \mathcal{X}_{et} . Recall (2.7) that, for any n , $M \in \text{Mod}(\mathcal{X}, \Lambda_n)$ is Noetherian if and only if M is constructible.

Proposition 4.1. *Let $M, N \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be AR-adic. Suppose that N is normalized. Then $M \otimes_{\Lambda_{\bullet}}^L N$ is AR-adic.*

Proof. By 3.26 and 3.13 (ii), we may assume M normalized. Then, by 3.12 and 3.13 (i), $M \otimes_{\Lambda_\bullet}^L N$ belongs to D^+ and is normalized. By 3.27, M_n and N_n belongs to D_c for all n . Hence, by 2.8, $e_n^{-1}(M \otimes_{\Lambda_\bullet}^L N) \simeq M_n \otimes_{\Lambda_n}^L N_n$ belongs to D_c for all n . Therefore $M \otimes_{\Lambda_\bullet}^L N$ is AR-adic by 3.27. \square

For $M \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ AR-adic and $N \in D_c^+(\mathcal{X}, \mathcal{O})$, define

$$M \otimes_{\mathcal{O}}^L N = M \otimes_{\Lambda_\bullet}^L \hat{N}.$$

By 4.1 and 3.13, this gives a functor

$$- \otimes_{\mathcal{O}}^L -: D_c^+(\mathcal{X}, \mathcal{O}) \times D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O}).$$

Using this definition, for $N \in D_c^+(\mathcal{X}, \mathcal{O})$, the projection formula 3.6

$$(e_{n*} \Lambda_n) \otimes_{\Lambda_\bullet}^L \hat{N} \xrightarrow{\sim} e_{n*} e_n^{-1} \hat{N}$$

can be reformulated as

$$(e_{n*} \Lambda_n) \otimes_{\mathcal{O}}^L N \xrightarrow{\sim} e_{n*} (\Lambda_n \otimes_{\mathcal{O}}^L N).$$

Proposition 4.2. *Let $M \in D^-(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ be normalized and AR-adic, $N \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ be AR-adic. Then $R\mathcal{H}om_{\Lambda_\bullet}(M, N) \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ is AR-adic.*

Proof. Up to replacing M by $\widehat{\tau^{\geq a} M} \simeq L\pi^* \tau^{\geq a} R\pi_* M$ (3.11), we may assume $M \in D^b$. By 3.26 and 3.16, we may assume N normalized. Then, by 3.15, $R\mathcal{H}om_{\Lambda_\bullet}(M, N)$ is normalized. By 3.27, M_n and N_n belongs to D_c for all n . Hence, by 3.14 and 2.8, $e_n^{-1} R\mathcal{H}om_{\Lambda_\bullet}(M, N) \simeq R\mathcal{H}om_{\Lambda_n}(M_n, N_n)$ belongs to D_c . Therefore $R\mathcal{H}om_{\Lambda_\bullet}(M, N)$ is AR-adic by 3.27. \square

For $M \in D_c^b(\mathcal{X}, \mathcal{O})$ and $N \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ AR-adic, define

$$R\mathcal{H}om_{\mathcal{O}}(M, N) = R\mathcal{H}om_{\Lambda_\bullet}(\hat{M}, N).$$

By 4.2 and 3.16, this gives a functor

$$R\mathcal{H}om_{\mathcal{O}}(-, -): D_c^b(\mathcal{X}, \mathcal{O})^{\text{op}} \times D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O}).$$

Proposition 4.3. (i) *For all $M, N \in D_c^+(\mathcal{X}, \mathcal{O})$, the map*

$$\Lambda_n \otimes_{\mathcal{O}}^L (M \otimes_{\mathcal{O}}^L N) \simeq (\Lambda_n \otimes_{\mathcal{O}}^L M) \otimes_{\Lambda_n} (\Lambda_n \otimes_{\mathcal{O}}^L N)$$

is an isomorphism.

(ii) *For all $M \in D_c^b(\mathcal{X}, \mathcal{O})$, $N \in D_c^+(\mathcal{X}, \mathcal{O})$, the map*

$$\Lambda_n \otimes_{\mathcal{O}}^L R\mathcal{H}om_{\mathcal{O}}(M, N) \rightarrow R\mathcal{H}om_{\Lambda_n}(\Lambda_n \otimes_{\mathcal{O}}^L M, \Lambda_n \otimes_{\mathcal{O}}^L N)$$

is an isomorphism.

Proof. (i) By 3.26, we may assume that $M, N \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ are normalized. Then $M \otimes_{\Lambda_\bullet}^L N$ is normalized by 3.12 and the assertion is trivial.

(ii) By 3.26, we may assume that $M \in D^b(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ and $N \in D^+(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ are normalized. Then, by 3.15, $R\mathcal{H}om_{\Lambda_\bullet}(M, N)$ is normalized. Hence the assertion follows from 3.14. \square

Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford stacks, \mathcal{Y} of finite type over S , \mathcal{X} of finite type over a regular scheme S' of dimension ≤ 1 . It induces a flat morphism of ringed toposes $(f_*^\mathbb{N}, f^{*\mathbb{N}}): (\mathcal{X}_{\text{et}}^\mathbb{N}, \Lambda_\bullet) \rightarrow (\mathcal{Y}_{\text{et}}^\mathbb{N}, \Lambda_\bullet)$. The functor $f^{*\mathbb{N}}: D(\mathcal{Y}^\mathbb{N}, \Lambda_\bullet) \rightarrow D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$ preserves AR-null complexes and AR-adic complexes. It induces $f^*: D_c^+(\mathcal{Y}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O})$ of cohomological dimension 0. For $N \in D_c^+(\mathcal{Y}, \mathcal{O})$, the map

$$f^*(\Lambda_n \otimes_{\mathcal{O}}^L N) \rightarrow \Lambda_n \otimes_{\mathcal{O}}^L f^* N$$

is an isomorphism. In fact, the map $f^{*\mathbb{N}} \hat{N} \rightarrow \widehat{f^* N}$ is an isomorphism.

Assume $S' = S$. The functor $Rf_*^\mathbb{N}: D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet) \rightarrow D(\mathcal{Y}^\mathbb{N}, \Lambda_\bullet)$ preserves AR-null complexes in D^+ by 3.7.

Proposition 4.4. *The functor Rf_* preserves AR-adic complexes in D^+ . If f has prime to ℓ inertia, then $f_*^{\mathbb{N}}$ has finite cohomological dimension.*

In particular, $Rf_*^{\mathbb{N}}$ induces $Rf_*: D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{Y}, \mathcal{O})$. If f has prime to ℓ inertia, then $Rf_*^{\mathbb{N}}$ preserves small direct sums [KS06, 14.3.4 (ii)] and Rf_* sends $D_c^b(\mathcal{X}, \mathcal{O})$ to $D_c^b(\mathcal{X}, \mathcal{O})$.

Proof. Let $M \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ be AR-adic. To prove that Rf_*M is AR-adic, we may assume M normalized. Then $Rf_*^{\mathbb{N}}M$ is normalized by 3.18. By 3.27, M_n belongs to D_c for all n . Hence, by 3.7 and 2.9 (i), $e_n^{-1}Rf_*^{\mathbb{N}}M \simeq Rf_*M_n$ belongs to D_c for all n . Therefore $Rf_*^{\mathbb{N}}M$ is AR-adic by 3.27. The last assertion follows from 3.7 and the fact that the cohomological dimension of $f_*: \text{Mod}(\mathcal{X}, \Lambda_n) \rightarrow \text{Mod}(\mathcal{Y}, \Lambda_n)$ does not depend on n . \square

Proposition 4.5. *For all $M \in D_c^+(\mathcal{X}, \mathcal{O})$, the map*

$$\Lambda_n \otimes_{\mathcal{O}}^L Rf_*M \rightarrow Rf_*(\Lambda_n \otimes_{\mathcal{O}}^L M)$$

is an isomorphism.

Proof. By 3.26, we may assume that $M \in D^+(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is normalized. Then, by 3.18, Rf_*M is normalized. Hence the assertion follows from 3.7. \square

Proposition 4.6. (i) (Base change) *Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-Cartesian square of Deligne-Mumford stacks with \mathcal{X} and \mathcal{Y} of finite type over S . Assume that one of the following conditions holds:

(a) *f is proper and \mathcal{Y}' is of finite type over a regular scheme S' of dimension ≤ 1 .*

(b) *g is smooth and of finite type.*

Then the base change map

$$g^*Rf_*M \rightarrow Rf'_*h^*M$$

is an isomorphism for all $M \in D_c^+(\mathcal{X}, \mathcal{O})$.

(ii) (Projection formula) *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of Deligne-Mumford S -stacks of finite type, $M \in D_c^+(\mathcal{X}, \mathcal{O})$, $N \in D_c^+(\mathcal{Y}, \mathcal{O})$. Then the map*

$$N \otimes_{\mathcal{O}}^L Rf_*M \rightarrow Rf_*(f^*N \otimes_{\mathcal{O}}^L M)$$

is an isomorphism.

Proof. (i) By 3.7, the proper or smooth (1.7) base change map $g^{\mathbb{N}*}Rf_*^{\mathbb{N}}M \rightarrow Rf_*^{\mathbb{N}}h^{\mathbb{N}*}M$ is an isomorphism.

(ii) By (i), we may assume that \mathcal{Y} is the spectrum of a separably closed field. As in the proof of 3.26, we may assume $\hat{N} \simeq L\pi^*K$, where K is a finite \mathcal{O} -module. Replacing K by a finite free resolution, the assertion becomes clear. \square

Proposition 4.7. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford stacks, \mathcal{Y} of finite type over S . Let $M \in D_c^b(\mathcal{Y}, \mathcal{O})$, $L \in D_c^+(\mathcal{Y}, \mathcal{O})$. Assume that one of the following conditions holds:*

(a) *The cohomology sheaves of M are lisse (3.23) and \mathcal{X} is of finite type over some regular scheme S' of dimension ≤ 1 .*

(b) f is smooth and of finite type.

Then the map

$$f^* R\mathcal{H}om_{\mathcal{Y}}(M, L) \rightarrow R\mathcal{H}om_{\mathcal{X}}(f^* M, f^* L)$$

is an isomorphism.

Proof. (a) As in the proof of 3.26, we may assume that $M = \hat{F}$, where $F \in \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ is adic and either flat or annihilated by λ^{n+1} for some n . In both cases, F is locally constant and it follows that the cohomology sheaves of $e_m^{-1} M$ are locally constant. The assertion then follows from 3.14.

(b) We may assume M and L normalized. The assertion then follows from 3.14 and 1.8. \square

Let $p: \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of Deligne-Mumford S -stacks of inertia type. If p has prime to ℓ inertia and the fibers of p have dimension $\leq d$, then $p_*^{\mathbb{N}}$ has cohomological dimension $\leq 2d$.

Lemma 4.8. *Let $p: \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of prime to ℓ inertia of Deligne-Mumford S -stacks of finite type. For all $M \in D(\mathcal{X}, \Lambda_n)$, the map (3.7.2)*

$$Le_n! Rp_* M \rightarrow Rp_*^{\mathbb{N}} Le_n! M$$

is an isomorphism.

Proof. By 3.7, it suffices to show that the map

$$\Lambda_q \otimes_{\Lambda_n}^L Rp_* M \rightarrow Rp_*(\Lambda_q \otimes_{\Lambda_n}^L M)$$

is an isomorphism for $q \leq n$, which is projection formula 1.15 (ii). \square

4.9. Let $j: \mathcal{U} \rightarrow \mathcal{X}$ be an étale representable morphism of finite type Deligne-Mumford S -stacks. Then 3.19 applies and we have $j_!^{\mathbb{N}}: D(\mathcal{U}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. It follows from (3.19.1) and projection formula (1.3.3) that $j_!^{\mathbb{N}}$ preserves AR-null and normalized complexes.

4.10. We construct $Rf_!^{\mathbb{N}}$ by gluing as in 1.17. Let \mathcal{C} be the 2-category of Deligne-Mumford S -stacks of finite type and finite inertia whose 1-cells are the separated morphisms of prime to ℓ inertia, \mathcal{A} be the 2-faithful subcategory whose 1-cells are the open immersions, \mathcal{B} be the 2-faithful subcategory whose 1-cells are the proper morphisms, \mathcal{D} be the 2-category of triangulated categories. Let $F_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{D}$ be the pseudo-functor

$$\mathcal{X} \mapsto D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}), \quad j \mapsto j_!^{\mathbb{N}}, \quad \alpha \mapsto \alpha_!^{\mathbb{N}},$$

and $F_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{D}$ be the pseudo-functor

$$\mathcal{X} \mapsto D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}), \quad p \mapsto Rp_*^{\mathbb{N}}, \quad \alpha \mapsto R\alpha_*^{\mathbb{N}}.$$

If f is a proper open immersion, let $\rho(f)$ be the inverse of the 2-cell (1.5.1) $f_!^{\mathbb{N}} \Rightarrow Rf_*^{\mathbb{N}}$. If D is a 2-Cartesian square (7.3.1), let $G_D: i_!^{\mathbb{N}} Rq_*^{\mathbb{N}} \Rightarrow Rp_*^{\mathbb{N}} j_!^{\mathbb{N}}$ be the 2-cell as in (1.4.2). By 7.23, $(F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho)$ is an object of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. By 1.13, it defines a pseudo-functor $F: \mathcal{C} \rightarrow \mathcal{D}$. For any 1-cell of \mathcal{C} , define

$$Rf_!^{\mathbb{N}}: D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D(\mathcal{Y}^{\mathbb{N}}, \Lambda_{\bullet})$$

to be $F(f)$. If the fibers of f have dimension $\leq d$, then $Rf_!^{\mathbb{N}}$ has cohomological amplitude contained in $[0, 2d]$. If f is representable and étale, then $Rf_!^{\mathbb{N}}$ coincides with $f_!^{\mathbb{N}}$ (4.9) as in 1.21.

We construct the support-forgetting map $Rf_!^{\mathbb{N}} \Rightarrow Rf_*^{\mathbb{N}}$, the base change isomorphism and the projection formula isomorphism by gluing as before. We construct two more isomorphisms by gluing. We define a pseudo-natural transformation $\epsilon: F \rightarrow F_n$ with $\epsilon(\mathcal{X}) = e_n^*$ by gluing the inverse of (3.19.1) for \mathcal{A} and (3.7.1) for \mathcal{B} , which is possible by 7.24 and 7.25. We define a pseudo-natural transformation $\eta: F_n \rightarrow F$ with $\eta(\mathcal{X}) = Le_n!$ by gluing the inverse of (3.19.2) for \mathcal{A} and 4.8 for \mathcal{B} , which is possible by 7.23. We have obtained the following.

Lemma 4.11. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of prime to ℓ inertia of Deligne-Mumford stacks of finite type and finite inertia. Let $M \in D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, $N \in D(\mathcal{Y}^{\mathbb{N}}, \Lambda_{\bullet})$. Then the maps*

$$(4.11.1) \quad e_n^{-1} Rf_{!}^{\mathbb{N}} M \rightarrow Rf_{!}^{\mathbb{N}} e_n^{-1} M,$$

$$(4.11.2) \quad Le_{n!} Rf_{!} N \rightarrow Rf_{!}^{\mathbb{N}} Le_{n!} N$$

are isomorphisms.

It follows from (4.11.1) that $Rf_{!}^{\mathbb{N}}$ preserves AR-null complexes. It follows from (4.11.1) and projection formula 1.18 (ii) that $Rf_{!}^{\mathbb{N}}$ preserves normalized complexes in D^{+} . It also follows from (4.11.1) that $Rf_{!}^{\mathbb{N}}$ commutes with small direct sums, hence, by [KS06, 14.3.1 (ix)], admits a right adjoint

$$Rf^{\mathbb{N}!}: D(\mathcal{Y}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}).$$

4.12. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prime to ℓ inertia of Deligne-Mumford S -stacks of finite type. Assume either (a) f is representable and étale; or (b) f is separated and \mathcal{X}, \mathcal{Y} are of finite inertia. Then $Rf_{!}^{\mathbb{N}}$ is defined in 4.9 and 4.10. We prove as in 4.4 that it preserves AR-adic complexes. It induces

$$Rf_{!}: D_c^{+}(\mathcal{X}, \mathcal{O}) \rightarrow D_c^{+}(\mathcal{Y}, \mathcal{O}),$$

which sends D_c^b to D_c^b .

4.13. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford S -stacks of finite type and finite inertia. We define

$$Rf_{\dagger}^{\mathbb{N}}: D^{+}(\mathcal{Y}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D^{+}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$$

by gluing $j_{!}^{\mathbb{N}}$ for j an open immersion and $Rp_{*}^{\mathbb{N}}$ for p proper as in 1.25. If f is of prime to ℓ inertia, then we obtain the restriction of $Rf_{!}^{\mathbb{N}}$ to D^{+} . We construct the support-forgetting map $Rf_{\dagger}^{\mathbb{N}} \Rightarrow Rf_{*}^{\mathbb{N}}$ and the base change isomorphism by gluing as before. Moreover, we glue (3.7.1) and the inverse of (3.19.1) and obtain the following.

Lemma 4.14. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford S -stacks of finite type and finite inertia, $M \in D^{+}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$. Then the map*

$$e_n^{-1} Rf_{\dagger}^{\mathbb{N}} M \rightarrow Rf_{\dagger}^{\mathbb{N}} e_n^{-1} M$$

is an isomorphism.

In particular, $Rf_{\dagger}^{\mathbb{N}}$ preserves AR-null complexes. It follows from 3.18 and projection formula (1.3.3) that $Rf_{\dagger}^{\mathbb{N}}$ preserves normalized complexes. One checks as in 4.4 that it preserves AR-adic complexes. Thus it induces

$$Rf_{\dagger}: D_c^{+}(\mathcal{X}, \mathcal{O}) \rightarrow D_c^{+}(\mathcal{Y}, \mathcal{O}),$$

endowed with a support-forgetting map $Rf_{\dagger} \Rightarrow Rf_{*}$ which is a natural equivalence if f is proper. If f is of prime to ℓ inertia, then Rf_{\dagger} is simply $Rf_{!}$.

4.15. Let $i: \mathcal{Y} \rightarrow \mathcal{X}$ be a closed immersion of finite type Deligne-Mumford S -stacks. The functor $i^{\mathbb{N}!}: \text{Mod}(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow \text{Mod}(\mathcal{Y}^{\mathbb{N}}, \Lambda_{\bullet})$ is a right adjoint of $i_{*}^{\mathbb{N}}$, and thus is left exact. Let $Ri^{\mathbb{N}!}: D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet}) \rightarrow D(\mathcal{Y}^{\mathbb{N}}, \Lambda_{\bullet})$ be its right derived functor. This is compatible with the definition following 4.11.

Let $j: \mathcal{U} \rightarrow \mathcal{X}$ be the complementary open immersion. For any complex M of Λ_{\bullet} -modules on $\mathcal{X}^{\mathbb{N}}$, we have a natural short exact sequence

$$0 \rightarrow j_{!}^{\mathbb{N}} j^{\mathbb{N}*} M \rightarrow M \rightarrow i_{*}^{\mathbb{N}} i^{\mathbb{N}*} M \rightarrow 0,$$

hence a distinguished triangle in $D(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$

$$(4.15.1) \quad j_{!}^{\mathbb{N}} j^{\mathbb{N}*} M \rightarrow M \rightarrow i_{*}^{\mathbb{N}} i^{\mathbb{N}*} M \rightarrow .$$

For any complex N of injective Λ_\bullet -modules on $\mathcal{X}^\mathbb{N}$, we have a natural short exact sequence

$$0 \rightarrow i_*^{\mathbb{N}} i^{\mathbb{N}} N \rightarrow N \rightarrow j_*^{\mathbb{N}} j^{\mathbb{N}*} N \rightarrow 0,$$

hence, for any $N \in D(\mathcal{X}^\mathbb{N}, \Lambda_\bullet)$, a distinguished triangle

$$(4.15.2) \quad i_*^{\mathbb{N}} R i^{\mathbb{N}!} N \rightarrow N \rightarrow R j_*^{\mathbb{N}} j^{\mathbb{N}*} N \rightarrow .$$

Combining the isomorphisms induced from 4.8 and (4.11.2) by adjunction, we have the following.

Lemma 4.16. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of prime to ℓ inertia of Deligne-Mumford S -stacks of finite type. Assume either f is a closed immersion, or \mathcal{X} and \mathcal{Y} are of finite inertia. Then, for all $M \in D(\mathcal{Y}^\mathbb{N}, \Lambda_\bullet)$, the map*

$$e_n^{-1} R f^{\mathbb{N}!} M \rightarrow R f^! e_n^{-1} M$$

is an isomorphism.

It follows that $R f^{\mathbb{N}!}$ preserves AR-null complexes.

Proposition 4.17. *In the situation of 4.16, $R f^{\mathbb{N}!}$ preserves normalized complexes in D^+ .*

Proof. The problem being local for the étale topology on \mathcal{X} and \mathcal{Y} , we reduce to two cases: (a) f is a closed immersion; (b) f is a smooth morphism of schemes. Case (a) follows from the distinguished triangle (4.15.2). Case (b) follows from the fact that $R f^{\mathbb{N}!} \simeq f^{\mathbb{N}*}(d)[2d]$, where d is the relative dimension of f . \square

One checks as in 4.4 that $R f^{\mathbb{N}!}$ preserves AR-adic complexes. It induces

$$R f^!: D_c^+(\mathcal{Y}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O}),$$

which sends D_c^b to D_c^b .

As in 4.5, we have the following.

Proposition 4.18. *Let f be a morphism of Deligne-Mumford S -stacks of finite type, $M \in D_c^+(\mathcal{X}, \mathcal{O})$, $N \in D_c^+(\mathcal{Y}, \mathcal{O})$.*

(i) If f is representable and étale, then the map

$$\Lambda_n \otimes_{\mathcal{O}}^L R f_! M \rightarrow R f_! (\Lambda_n \otimes_{\mathcal{O}}^L M)$$

is an isomorphism.

(ii) If f is separated, \mathcal{X} and \mathcal{Y} are of finite inertia, then the map

$$\Lambda_n \otimes_{\mathcal{O}}^L R f_{\dagger} M \rightarrow R f_{\dagger} (\Lambda_n \otimes_{\mathcal{O}}^L M)$$

is an isomorphism.

(iii) If f is as in 4.16, then the map

$$\Lambda_n \otimes_{\mathcal{O}}^L R f^! N \rightarrow R f^! (\Lambda_n \otimes_{\mathcal{O}}^L N)$$

is an isomorphism.

4.19. Let $i: \mathcal{Y} \rightarrow \mathcal{X}$ be a closed immersion of Deligne-Mumford S -stacks of finite type, $j: \mathcal{U} \rightarrow \mathcal{X}$ be the complementary open immersion. For $M \in D_c^+(\mathcal{X}, \mathcal{O})$, (4.15.1) and (4.15.2) induce distinguished triangles

$$\begin{aligned} j_! j^* M &\rightarrow M \rightarrow i_* i^* M \rightarrow, \\ i_* R i^! M &\rightarrow M \rightarrow R j_* j^* M \rightarrow. \end{aligned}$$

Theorem 4.20. (i) (Base change) Let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-Cartesian square of Deligne-Mumford stacks, f separated, \mathcal{X} and \mathcal{Y} of finite type and finite inertia over S , \mathcal{Y}' of finite type and finite inertia over some regular scheme S' of dimension ≤ 1 . Then, for all $M \in D_c^+(\mathcal{X}, \mathcal{O})$, the map

$$g^* Rf_+ M \rightarrow Rf'_+ h^* M$$

is an isomorphism.

(ii) (Projection formula) Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of Deligne-Mumford S -stacks of finite type and finite inertia, $M \in D_c^+(\mathcal{X}, \mathcal{O})$, $N \in D_c^+(\mathcal{Y}, \mathcal{O})$. Then the map

$$N \otimes_{\mathcal{O}}^L Rf_+ M \rightarrow Rf_+(f^* N \otimes_{\mathcal{O}}^L M)$$

is an isomorphism.

Proof. (i) This follows from the construction 4.13.

(ii) By (i) we may assume that \mathcal{Y} the spectrum of a separably closed field. We may assume furthermore $N = L\pi^* F$, F an \mathcal{O} -module of finite type. It then suffices to take a finite free resolution of F . \square

The base change isomorphism constructed in 4.10 induces the following by adjunction.

Proposition 4.21. Let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-Cartesian square of Deligne-Mumford S -stacks of finite type and finite inertia, g separated of prime to ℓ inertia. Let $M \in D_c^+(\mathcal{X}, \mathcal{O})$. Then the map

$$Rf'_* Rh^! M \rightarrow Rg^! Rf_* M$$

is an isomorphism.

The isomorphisms (1.3.2), (1.23.1), (1.6.3) and (1.23.2) induce the following.

Proposition 4.22. Let f be a morphism of Deligne-Mumford S -stacks of finite type, $K \in D_c^b(\mathcal{X}, \mathcal{O})$, $L \in D_c^+(\mathcal{Y}, \mathcal{O})$, $M \in D_c^b(\mathcal{Y}, \mathcal{O})$.

(i) If f is representable and étale, then the map

$$R\mathcal{H}om_{\mathcal{Y}}(Rf_! K, L) \rightarrow Rf_* R\mathcal{H}om_{\mathcal{X}}(K, f^* L)$$

is an isomorphism.

(ii) If f is separated of prime to ℓ inertia, and \mathcal{X} and \mathcal{Y} are of finite inertia, then the map

$$R\mathcal{H}om_{\mathcal{Y}}(Rf_! K, L) \rightarrow Rf_* R\mathcal{H}om_{\mathcal{X}}(K, Rf^! L)$$

is an isomorphism.

(iii) If f is as in 4.16, then the map

$$R\mathcal{H}om_{\mathcal{X}}(f^* M, Rf^! L) \rightarrow Rf^! R\mathcal{H}om_{\mathcal{Y}}(M, L)$$

is an isomorphism.

4.23. Let $\Omega_S \in D_{AR-a}^b(S^{\mathbb{N}}, \Lambda_{\bullet})$ be the object such that $\Omega_S|_T = \Lambda_{\bullet}(d_T)[2d_T]$ for every connected component T of S , where $d_T = \dim T$. Then Ω_S is clearly normalized. For any scheme X separated of finite type over S , let $\Omega_X = Ra^{\mathbb{N}}\Omega_S \in D^b(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$, where $a: X \rightarrow S$ is the structural morphism. Then Ω_X is AR-adic and normalized. It follows that $R\mathcal{H}om_{\Lambda_{\bullet}}(\Omega_X, \Omega_X) \in D^{\geq 0}$ and $\text{Ext}_{\Lambda_{\bullet}}^i(\Omega_X, \Omega_X) = 0$ for $i < 0$.

Let \mathcal{X} be a Deligne-Mumford S -stack of finite type. For every étale morphism $\alpha: X \rightarrow \mathcal{X}$ with X a scheme separated of finite type over S , let $h_{\alpha} \in \mathcal{X}_{\text{et}}$ be the sheaf represented by α and $h_{\alpha, \bullet} \in \mathcal{X}_{\text{et}}^{\mathbb{N}}$ be the constant projective system. Then $\mathcal{X}_{\text{et}}^{\mathbb{N}}/h_{\alpha, \bullet}$ can be identified with $X_{\text{et}}^{\mathbb{N}}$. For such $\alpha: X \rightarrow \mathcal{X}$, we associate Ω_X . For any morphism $f: X \rightarrow Y$ between such morphisms, there is a canonical isomorphism $f^*\Omega_Y \xrightarrow{\sim} \Omega_X$. Since any finite product of $h_{\alpha, \bullet}$'s can be covered by a family of $h_{\beta, \bullet}$'s, we obtain an object of $D^{[-2d_{\mathcal{X}}, 0]}$ given C -locally, where C is the sieve generated by the $h_{\alpha, \bullet}$'s, $d_{\mathcal{X}} = \max_{s \in S}(\dim \mathcal{X}_s + d_s)$, d_s is the dimension of the closure of s in S . By the “BBD gluing lemma” [BBD82, 3.2.4], there exists a unique $\Omega_{\mathcal{X}} \in D^b(\mathcal{X}^{\mathbb{N}}, \Lambda_{\bullet})$ such that, for all étale morphism $\alpha: X \rightarrow \mathcal{X}$ with X a separated finite type S -scheme, we have $\alpha^*\Omega_{\mathcal{X}} \simeq \Omega_X$. It follows that $\Omega_{\mathcal{X}}$ is AR-adic and normalized. Thus

$$\Lambda_n \otimes_{\mathcal{O}}^L \Omega_{\mathcal{X}} \simeq e_n^{-1} \Omega_{\mathcal{X}} \simeq \Omega_{\mathcal{X}, n}.$$

Define a triangulated functor $D_{\mathcal{X}}: D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, \mathcal{O})$ by $D_{\mathcal{X}}M = R\mathcal{H}om_{\mathcal{O}}(M, \Omega_{\mathcal{X}})$.

Proposition 4.24. *Let \mathcal{X} be a finite type Deligne-Mumford S -stack. The functor $D_{\mathcal{X}}$ induces a reflexive functor $D_c^b(\mathcal{X}, \mathcal{O})^{\text{op}} \rightarrow D_c^b(\mathcal{X}, \mathcal{O})$ of cohomological amplitude contained in $[-2d_{\mathcal{X}}, 1]$. Moreover, for $M \in D_c^b(\mathcal{X}, \mathcal{O})$, the morphism*

$$\Lambda_n \otimes_{\mathcal{O}}^L D_{\mathcal{X}}M \rightarrow D_{\mathcal{X}, n}(\Lambda_n \otimes_{\mathcal{O}}^L M)$$

is an isomorphism.

Proof. The last assertion follows from 4.3. For $M \in D_c^{[a, b]}(\mathcal{X}, \mathcal{O})$, $\Lambda_n \otimes_{\mathcal{O}}^L M \in D_c^{[a-1, b]}(\mathcal{X}, \Lambda_n)$. By the last assertion and 2.12 (i), $\Lambda_n \otimes_{\mathcal{O}}^L D_{\mathcal{X}}M$ belongs to $D^{[-b-2d_{\mathcal{X}}, -a+1]}$ for all n . Hence $D_{\mathcal{X}}M$ belongs to $D_c^{[-b-2d_{\mathcal{X}}, -a+1]}(\mathcal{X}, \mathcal{O})$. Moreover, for all n , the following diagram commutes

$$\begin{array}{ccc} \Lambda_n \otimes_{\mathcal{O}}^L M & \longrightarrow & D_{\mathcal{X}, n} D_{\mathcal{X}, n}(\Lambda_n \otimes_{\mathcal{O}}^L M) \\ \downarrow & & \downarrow \simeq \\ \Lambda_n \otimes_{\mathcal{O}}^L D_{\mathcal{X}} D_{\mathcal{X}} M & \xrightarrow{\simeq} & D_{\mathcal{X}, n}(\Lambda_n \otimes_{\mathcal{O}}^L D_{\mathcal{X}} M) \end{array}$$

By 2.12 (ii), the top arrow is an isomorphism. Therefore, the map $M \rightarrow D_{\mathcal{X}} D_{\mathcal{X}} M$ is an isomorphism. \square

4.25. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a separated morphism of prime to ℓ inertia of Deligne-Mumford S -stacks of finite type. Assume either f is a closed immersion, or \mathcal{X} and \mathcal{Y} are of finite inertia. The isomorphism 4.22 (iii) gives an isomorphism $D_{\mathcal{X}} f^* \xrightarrow{\sim} Rf^! D_{\mathcal{Y}}$ of functors $D_c^b(\mathcal{Y}, \mathcal{O}) \rightarrow D_c^b(\mathcal{X}, \mathcal{O})$. Using biduality (4.24), we obtain an isomorphism of functors $D_c^b(\mathcal{Y}, \mathcal{O}) \rightarrow D_c^b(\mathcal{X}, \mathcal{O})$:

$$Rf^! \simeq Rf^! D_{\mathcal{Y}} D_{\mathcal{Y}} \simeq D_{\mathcal{X}} f^* D_{\mathcal{X}}.$$

4.26. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Deligne-Mumford S -stacks. Thanks to 4.24, we can define a triangulated functor

$$Rf^!: D_c^b(\mathcal{Y}, \mathcal{O}) \rightarrow D_c^b(\mathcal{X}, \mathcal{O}), \quad N \mapsto D_{\mathcal{X}} f^* D_{\mathcal{Y}} N.$$

By 4.25, this definition is compatible with previous definitions. For any $N \in D_c^b(\mathcal{Y}, \mathcal{O})$, 4.5 and 4.24 induce an isomorphism

$$\Lambda_n \otimes_{\mathcal{O}}^L Rf^! N \rightarrow Rf^! (\Lambda_n \otimes_{\mathcal{O}}^L N).$$

If $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ are two such morphisms, then we have an isomorphism of functors $D_c^b(\mathcal{Z}, \mathcal{O}) \rightarrow D_c^b(\mathcal{X}, \mathcal{O})$:

$$R(gf)^! = D_{\mathcal{X}}(gf)^* D_{\mathcal{Z}} \simeq D_{\mathcal{X}} f^* g^* D_{\mathcal{Z}} \simeq D_{\mathcal{X}} f^* D_{\mathcal{Y}} D_{\mathcal{Y}} g^* D_{\mathcal{Z}} = Rf^! Rg^!.$$

If f is smooth, it follows from the construction of $\Omega_{\mathcal{X}}$ and $\Omega_{\mathcal{Y}}$ that $\Omega_{\mathcal{X}} \simeq f^* \Omega_{\mathcal{Y}}(d)[2d]$, where d is the relative dimension of f . It follows that 4.7 induces an isomorphism of functors $f^*(D_{\mathcal{Y}} N)(d)[2d] \simeq D_{\mathcal{X}} f^* N$ for all $N \in D_c^b(\mathcal{Y}, \mathcal{O})$. Thus

$$Rf_! = D_{\mathcal{X}} f^* D_{\mathcal{Y}} \simeq f^*(d)[2d].$$

4.27. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prime to ℓ inertia between Deligne-Mumford S -stacks of finite type. Assume either (a) f is representable and étale, or (b) f is separated and \mathcal{X} and \mathcal{Y} are of finite inertia. The isomorphisms 4.22 (i) and (ii) give an isomorphism $D_{\mathcal{Y}} Rf_! \xrightarrow{\sim} Rf_* D_{\mathcal{X}}$ of functors $D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^b(\mathcal{Y}, \mathcal{O})$. Using biduality (4.24), we obtain an isomorphism of functors $D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^b(\mathcal{Y}, \mathcal{O})$:

$$Rf_! \simeq D_{\mathcal{Y}} D_{\mathcal{Y}} Rf_! \simeq D_{\mathcal{Y}} Rf_* D_{\mathcal{X}}.$$

4.28. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prime to m inertia between Deligne-Mumford S -stacks of finite type. Thanks to 4.24, we can define a triangulated functor

$$Rf_!: D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^b(\mathcal{Y}, \mathcal{O}), \quad M \mapsto D_{\mathcal{Y}} Rf_* D_{\mathcal{X}} M.$$

By 4.27, this definition is compatible with previous definitions. For $M \in D_c^b(\mathcal{X}, \mathcal{O})$, 4.5 and 4.24 induce an isomorphism

$$\Lambda_n \otimes_{\mathcal{O}}^L Rf_! M \rightarrow Rf_! (\Lambda_n \otimes_{\mathcal{O}}^L M).$$

If $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ are two such morphisms, then we have an isomorphism of functors $D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^b(\mathcal{Y}, \mathcal{O})$:

$$R(gf)_! = D_{\mathcal{Z}} R(gf)_* D_{\mathcal{X}} \simeq D_{\mathcal{Z}} Rg_* Rf_* D_{\mathcal{X}} \simeq D_{\mathcal{Z}} Rg_* D_{\mathcal{Y}} D_{\mathcal{Y}} Rf_* D_{\mathcal{X}} = Rg_! Rf_!.$$

5 Variants and complements

In 5.1 through 5.6, let S be a regular scheme of dimension ≤ 1 .

5.1. Let E be a discrete valuation field of characteristic 0 and residue characteristic ℓ invertible on S , let \mathcal{O} be its ring of integers. For \mathcal{X} a finite type Deligne-Mumford S -stack, let \mathcal{T} be the full subcategory of $\text{Mod}_c(\mathcal{X}, \mathcal{O})$ consisting of sheaves annihilated by some power of ℓ . Then \mathcal{T} is a thick subcategory and we define $\text{Mod}_c(\mathcal{X}, E)$ to be the quotient $\text{Mod}_c(\mathcal{X}, \mathcal{O})/\mathcal{T}$. We have $\text{Mod}_c(\mathcal{X}, E) \simeq \text{Mod}_c(\mathcal{X}, \mathcal{O}) \otimes_{\mathcal{O}} E$. Let $D_{\mathcal{T}}^+$ be the full subcategory of $D_c^+(\mathcal{X}, \mathcal{O})$ consisting of complexes whose cohomology sheaves belong to \mathcal{T} . Then $D_{\mathcal{T}}^+$ is a thick triangulated subcategory and we define $D_c^+(\mathcal{X}, E)$ to be the quotient $D_c^+(\mathcal{X}, \mathcal{O})/D_{\mathcal{T}}^+$. The canonical t-structure on $D_c^+(\mathcal{X}, \mathcal{O})$ induces a canonical t-structure on $D_c^+(\mathcal{X}, E)$, which has heart $\text{Mod}_c(\mathcal{X}, E)$ and enables us to define $D^b(\mathcal{X}, E)$. We have $D_c^b(\mathcal{X}, E) \simeq D_c^b(\mathcal{X}, \mathcal{O}) \otimes_{\mathcal{O}} E$. We denote the image of M under the canonical functor $D_c^+(\mathcal{X}, \mathcal{O}) \rightarrow D_c^+(\mathcal{X}, E)$ by $E \otimes_{\mathcal{O}} M$.

The functors $- \otimes_{\mathcal{O}} -$, $R\mathcal{H}om_{\mathcal{O}}(-, -)$, $D_{\mathcal{X}}$ in § 4 induce functors

$$\begin{aligned} - \otimes_E -: D_c^+(\mathcal{X}, E) \times D_c^+(\mathcal{X}, E) &\rightarrow D_c^+(\mathcal{X}, E), \\ R\mathcal{H}om_E(-, -): D_c^b(\mathcal{X}, E)^{\text{op}} \times D_c^+(\mathcal{X}, E) &\rightarrow D_c^+(\mathcal{X}, E), \end{aligned}$$

and a reflexive triangulated functor

$$D_{\mathcal{X}}: D_c^b(\mathcal{X}, E)^{\text{op}} \rightarrow D_c^b(\mathcal{X}, E).$$

Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Deligne-Mumford S -stacks. The functors f^* and Rf_* in § 4 induce functors

$$f^*: D_c^+(\mathcal{Y}, E) \rightarrow D_c^+(\mathcal{X}, E), \quad Rf_*: D_c^+(\mathcal{X}, E) \rightarrow D_c^+(\mathcal{Y}, E).$$

If f is separated and \mathcal{X} and \mathcal{Y} are of finite inertia, then the functor $Rf_!$ in § 4 induces a functor

$$Rf_!: D_c^+(\mathcal{X}, E) \rightarrow D_c^+(\mathcal{Y}, E).$$

Proposition 5.2. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of finite type Deligne-Mumford S -stacks. Assume that the fibers of f have dimension $\leq d$. Then*

(i) *Rf_* sends $D_c^b(\mathcal{X}, E)$ to $D_c^b(\mathcal{Y}, E)$. Moreover, if f is proper, then the cohomological amplitude of Rf_* is contained in $[0, 2d]$.*

(ii) *If f is separated and \mathcal{X} and \mathcal{Y} are of finite inertia, then $Rf_!$ sends $D_c^b(\mathcal{X}, E)$ to $D_c^b(\mathcal{Y}, E)$ and has cohomological amplitude contained in $[0, 2d]$.*

Proof. (i) We may assume \mathcal{Y} is a scheme. For the first assertion, using reductions as in 1.27 (i), we may assume \mathcal{X} separated over S . Then, for both assertions, since f factors through the coarse space of \mathcal{X} , we are reduced to two cases: (a) f is a universal homeomorphism; (b) f is representable. Case (b) is known. In case (a), by proper base change, we may assume that \mathcal{Y} is the spectrum of an algebraically closed field. Then $\mathcal{X}^{\text{red}} \simeq BG$ and the assertions are clear.

(ii) By base change (4.20 (i)), we may assume that \mathcal{Y} is the spectrum of a field. The assertion then follows from the decomposition 1.10 and (i). \square

Proposition 5.3. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a universal homeomorphism (1.9) of Deligne-Mumford S -stacks of finite type, $L \in D_c^+(\mathcal{Y}, E)$. Then the adjunction map $L \rightarrow Rf_* f^* L$ is an isomorphism.*

Proof. By proper base change, we may assume that \mathcal{Y} is the spectrum of an algebraically closed field. Then $\mathcal{X}^{\text{red}} \simeq BG$ and the assertion is clear. \square

5.4. Let f be a morphism of Deligne-Mumford S -stacks of finite type. The functor $Rf^!$ in 4.26 induces a functor

$$Rf^!: D_c^b(\mathcal{Y}, E) \rightarrow D_c^b(\mathcal{X}, E).$$

The finiteness of Rf_* (5.2) enables us to define a functor

$$Rf_!: D_c^b(\mathcal{X}, E) \rightarrow D_c^b(\mathcal{Y}, E), \quad M \mapsto D_{\mathcal{Y}} Rf_* D_{\mathcal{X}} M.$$

If $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ are two such morphisms, then

$$R(gf)_! = D_{\mathcal{Z}} R(gf)_* D_{\mathcal{X}} \simeq D_{\mathcal{Z}} Rg_* Rf_* D_{\mathcal{X}} \simeq D_{\mathcal{Z}} Rg_* D_{\mathcal{Y}} D_{\mathcal{Y}} Rf_* D_{\mathcal{X}} = Rg_! Rf_!.$$

If f is separated and \mathcal{X} and \mathcal{Y} are of finite inertia, then $Rf_!$ is the restriction of $Rf_!$ to D_c^b . In fact, using the decomposition 1.10 of f , we are reduced to two cases: (a) f proper and quasi-finite, (b) f representable. In case (a), one can repeat the argument of [Ols08, 5.15]. Case (b) follows from 4.27.

5.5. Finally let E' be an algebraic extension of E . Define $\text{Mod}_c(\mathcal{X}, E')$ to be the 2-inductive limit of $\text{Mod}_c(\mathcal{X}, E'')$, $D_c^+(\mathcal{X}, E')$ to be the 2-inductive limit of $D_c^+(\mathcal{X}, E'')$, where E'' runs over all finite extensions of E contained in E' . Then $D_c^+(\mathcal{X}, E')$ admits the same operations as $D_c^+(\mathcal{X}, E)$.

5.6. Let \mathcal{O} be as in § 4, X be a scheme separated of finite type over S . In [Eke90, § 6], Ekedahl constructs a category $D_c^b(X, \mathcal{O})_{\text{Ek}}$ and the six operations on it. By 3.28 and [Eke90, 2.7 ii)], the normalization functor induces an equivalence of categories $D_c^b(X, \mathcal{O}) \rightarrow D_c^b(X, \mathcal{O})_{\text{Ek}}$. The six operations in § 4 and the six operations of Ekedahl are compatible via this equivalence.

5.7. Let S be a Noetherian quasi-excellent scheme of finite dimension. By Gabber's finiteness theorem [Org], the direct construction of the six operations in 4.1 through 4.22 and 5.1 through 5.3 can be carried out over S . Moreover, if S is excellent and admits a global dimension function, then Gabber's duality theorem allows one to carry out the construction by duality in §§ 2, 4 and 5.

6 Application to Brauer theory

Let \mathcal{O} be a complete discrete valuation ring with fraction field E of characteristic 0 and residue field F of characteristic $\ell > 0$, \mathfrak{m} be the maximal ideal of \mathcal{O} . Note that for any field F of characteristic $\ell > 0$, any Cohen ring of F satisfies the condition for \mathcal{O} . For any F -module (resp. \mathcal{O} -module, resp. E -module)

M of finite type, we endow M with the unique Hausdorff topology such that M is a topological module. For a profinite group G , we define a *coherent $F[G]$ -module* (resp. $\mathcal{O}[G]$ -module, resp. $E[G]$ -module) to be an F -module (resp. \mathcal{O} -module, resp. E -module) of finite type endowed with a continuous F -linear (resp. \mathcal{O} -linear, resp. E -linear) action of G , and we denote by $K(G, F)$ (resp. $K(G, \mathcal{O})$, resp. $K(G, E)$) the Grothendieck group of the category of such modules. Let $i: \operatorname{Spec} F \rightarrow \operatorname{Spec} \mathcal{O}$, $j: \operatorname{Spec} E \rightarrow \operatorname{Spec} \mathcal{O}$. Then we have homomorphisms $i_*: K(G, F) \rightarrow K(G, \mathcal{O})$ defined by restriction of scalars, and $j^*: K(G, \mathcal{O}) \rightarrow K(G, E)$ defined by extension of scalars $[M] \mapsto [E \otimes_{\mathcal{O}} M]$.

Lemma 6.1. *We have $i_* = 0$ and j^* is an isomorphism.*

Proof. We claim that the localization sequence

$$K(G, F) \xrightarrow{i_*} K(G, \mathcal{O}) \xrightarrow{j^*} K(G, E) \rightarrow 0$$

is exact. It is clear that $j^*i_* = 0$. Define a homomorphism $s: K(G, E) \rightarrow K(G, \mathcal{O})/\operatorname{Im} i_*$ by sending the class of an $E[G]$ -module M to the class of any G -stable \mathcal{O} -lattice L of M . If L_1 and L_2 are two G -stable lattices of M , then $[L_1] - [L_2]$ belongs to $\operatorname{Im} i_*$. Indeed, we may assume $L_1 \subset L_2$, in which case L_1/L_2 is killed by a power of \mathfrak{m} , hence its class belongs to $\operatorname{Im} i_*$. If $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is a short exact sequence of $E[G]$ -modules, L is a G -stable \mathcal{O} -lattice of M , then $f^{-1}(L)$ is a lattice of M and $g(L)$ is a lattice of M'' . Hence s is well-defined. It is clearly an inverse of the map $K(G, \mathcal{O})/\operatorname{Im} i_* \rightarrow K(G, E)$ induced by j^* .

It remains to show $i_* = 0$. By [Ser98, chap. 16, th. 33], the decomposition map $d: K(G, E) \rightarrow K(G, F)$, characterized by $d[j^*M] = [i^*M]$ for M torsion-free, is surjective. Hence it suffices to verify that for any torsion-free coherent $\mathcal{O}[G]$ -module M , we have $i_*[M/\mathfrak{m}M] = 0$, which is trivial. \square

Let S be a regular scheme of dimension ≤ 1 on which ℓ is invertible, \mathcal{X} be a finite type Deligne-Mumford S -stack \mathcal{X} . We denote by $K(\mathcal{X}, F)$ (resp. $K(\mathcal{X}, \mathcal{O})$, resp. $K(\mathcal{X}, E)$) the Grothendieck group of $\operatorname{Mod}_c(\mathcal{X}, F)$ (resp. $\operatorname{Mod}_c(\mathcal{X}, \mathcal{O})$, resp. $\operatorname{Mod}_c(\mathcal{X}, E)$). We denote by $i_*: K(\mathcal{X}, F) \rightarrow K(\mathcal{X}, \mathcal{O})$ the homomorphism induced by the exact restriction of scalars functor $e_{0*}: \operatorname{Mod}_c(\mathcal{X}, F) \rightarrow \operatorname{Mod}_c(\mathcal{X}, \mathcal{O})$, by $j^*: K(\mathcal{X}, \mathcal{O}) \rightarrow K(\mathcal{X}, E)$ the homomorphism induced by the exact functor (5.1)

$$\operatorname{Mod}_c(\mathcal{X}, \mathcal{O}) \rightarrow \operatorname{Mod}_c(\mathcal{X}, E), \quad M \mapsto E \otimes_{\mathcal{O}} M,$$

and by $i^*: K(\mathcal{X}, \mathcal{O}) \rightarrow K(\mathcal{X}, F)$ the homomorphism induced by the triangulated functor (3.28)

$$D_c^b(\mathcal{X}, \mathcal{O}) \rightarrow D_c^b(\mathcal{X}, F), \quad M \mapsto F \otimes_{\mathcal{O}}^L M.$$

Proposition 6.2. (a) $i_* = 0$ and j^* is an isomorphism.

(b) i^* is a surjection.

Proof. (a) We claim that the localization sequence

$$K(\mathcal{X}, F) \xrightarrow{i_*} K(\mathcal{X}, \mathcal{O}) \xrightarrow{j^*} K(\mathcal{X}, E) \rightarrow 0$$

is exact. Since $E \otimes_{\mathcal{O}} (e_{0*}M) = 0$ for all $M \in \operatorname{Mod}_c(\mathcal{X}, F)$, we have $j^*i_* = 0$. Define a homomorphism

$$s: K(\mathcal{X}, E) \rightarrow K(\mathcal{X}, \mathcal{O})/\operatorname{Im} i_*, \quad [E \otimes_{\mathcal{O}} M] \mapsto [M].$$

Note that if $M \in \operatorname{Mod}_c(\mathcal{X}, \mathcal{O})$ satisfies $E \otimes_{\mathcal{O}} M = 0$ in $\operatorname{Mod}_c(\mathcal{X}, E)$, then its class in $K(\mathcal{X}, \mathcal{O})$ belongs to $\operatorname{Im} i_*$. If

$$0 \rightarrow E \otimes_{\mathcal{O}} M' \xrightarrow{f} E \otimes_{\mathcal{O}} M \xrightarrow{g} E \otimes_{\mathcal{O}} M'' \rightarrow 0$$

is a short exact sequence in $\operatorname{Mod}_c(\mathcal{X}, E)$, then there exists an integer $n \geq 0$, $f_{\mathcal{O}}: M' \rightarrow M$ and $g_{\mathcal{O}}: M \rightarrow M''$ such that $\ell^n f = E \otimes_{\mathcal{O}} f_{\mathcal{O}}$ and $\ell^n g = E \otimes_{\mathcal{O}} g_{\mathcal{O}}$. Since $E \otimes_{\mathcal{O}} \operatorname{Ker} f_{\mathcal{O}}$, $E \otimes_{\mathcal{O}} (\operatorname{Im} f_{\mathcal{O}}/\operatorname{Im} f_{\mathcal{O}} \cap \operatorname{Ker} g_{\mathcal{O}})$, $E \otimes_{\mathcal{O}} (\operatorname{Ker} g_{\mathcal{O}}/\operatorname{Im} f_{\mathcal{O}} \cap \operatorname{Ker} g_{\mathcal{O}})$ and $E \otimes_{\mathcal{O}} \operatorname{Coker} g_{\mathcal{O}}$ are all zero objects of $\operatorname{Mod}_c(\mathcal{X}, E)$, it follows that

$[M'] - [M] + [M'']$ belongs to $\text{Im } v$. Hence s is well defined. It is clearly an inverse of the map $K(\mathcal{X}, \mathcal{O})/\text{Im } i_* \rightarrow K(\mathcal{X}, E)$ induced by j^* .

It remains to show $i_* = 0$. Note that the abelian group $K(\mathcal{X}, F)$ is generated by elements of the form $[f_! M]$, where $f: \mathcal{Y} \rightarrow \mathcal{X}$ is an immersion, \mathcal{Y} is integral, $M \in \text{Mod}_c(\mathcal{Y}, F)$ is lisse. For such f , we have a 2-commutative diagram

$$\begin{array}{ccc} \text{Mod}_{\text{lisse}}(\mathcal{Y}, F) & \xrightarrow{i_*} & \text{Mod}_{\text{lisse}}(\mathcal{Y}, \mathcal{O}) \\ \downarrow f_! & & \downarrow f_! \\ \text{Mod}_c(\mathcal{X}, F) & \xrightarrow{i_*} & \text{Mod}_c(\mathcal{X}, \mathcal{O}) \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc} K(G, F) & \xrightarrow{i_*} & K(G, \mathcal{O}) \\ \downarrow & & \downarrow \\ K(\mathcal{X}, F) & \xrightarrow{i_*} & K(\mathcal{X}, \mathcal{O}) \end{array}$$

where G is a fundamental group of \mathcal{Y} . We conclude by applying the fact $i_* = 0$ from Brauer theory (6.1).

(b) For f and G as before, we have a commutative diagram

$$\begin{array}{ccccc} K(G, \mathcal{O}) & \xrightarrow[\sim]{j^*} & K(G, E) & \xrightarrow{d} & K(G, F) \\ \downarrow & & & & \downarrow \\ K(\mathcal{X}, \mathcal{O}) & \xrightarrow{i^*} & & & K(\mathcal{X}, F) \end{array}$$

where the vertical maps are induced by $f_!$. It then suffices to apply the fact that d is surjective from Brauer theory. \square

6.3. Let us recall Laumon's theorem on Euler-Poincaré characteristics [IZ09, § 2]. We denote by $K^\sim(\mathcal{X}, E)$ the quotient of $K(\mathcal{X}, E)$ by the ideal generated by $[E(1)] - [E]$, and by x^\sim the image in $K^\sim(\mathcal{X}, E)$ of an element x of $K(\mathcal{X}, E)$. We denote by $K^\sim(\mathcal{X}, F)$ the quotient of $K(\mathcal{X}, F)$ by the ideal generated by $[F(1)] - [F]$, and by x^\sim the image in $K^\sim(\mathcal{X}, F)$ of an element x of $K(\mathcal{X}, F)$.

Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford S -stacks of finite type, the triangulated functors

$$Rf_*, Rf_!: D_c^b(\mathcal{X}, E) \rightarrow D_c^b(\mathcal{Y}, E)$$

induce homomorphisms

$$f_*, f_!: K(\mathcal{X}, E) \rightarrow K(\mathcal{Y}, E)$$

and

$$f_*, f_!: K^\sim(\mathcal{X}, E) \rightarrow K^\sim(\mathcal{Y}, E)$$

by passing to quotients.

If f is of prime to ℓ inertia, then the triangulated functors

$$Rf_*, Rf_!: D_c^b(\mathcal{X}, F) \rightarrow D_c^b(\mathcal{Y}, F)$$

induce homomorphisms

$$f_*, f_!: K(\mathcal{X}, F) \rightarrow K(\mathcal{Y}, F)$$

and

$$f_*, f_!: K^\sim(\mathcal{X}, F) \rightarrow K^\sim(\mathcal{Y}, F)$$

by passing to quotients.

Theorem 6.4. (a) For any $x \in K(\mathcal{X}, E)$, we have $f_*(x^\sim) = f_!(x^\sim)$ in $K^\sim(\mathcal{Y}, E)$.

(b) Assume f has prime to ℓ inertia. Then, for any $x \in K(\mathcal{X}, F)$, we have $f_*(x^\sim) = f_!(x^\sim)$ in $K^\sim(\mathcal{Y}, F)$.

In *loc. cit.*, the two assertions are proven using similar methods. With the help of 6.2, we can deduce (b) from (a) as follows. Let f be as in (b). We have a 2-commutative diagram of triangulated categories and triangulated functors (4.5)

$$\begin{array}{ccccc} D_c^b(\mathcal{X}, E) & \xleftarrow{E \otimes_{\mathcal{O}} -} & D_c^b(\mathcal{X}, \mathcal{O}) & \xrightarrow{F \otimes_{\mathcal{O}}^L -} & D_c^b(\mathcal{X}, F) \\ \downarrow Rf_* & & \downarrow Rf_* & & \downarrow Rf_* \\ D_c^b(\mathcal{Y}, E) & \xleftarrow{E \otimes_{\mathcal{O}} -} & D_c^b(\mathcal{Y}, \mathcal{O}) & \xrightarrow{F \otimes_{\mathcal{O}}^L -} & D_c^b(\mathcal{Y}, F) \end{array}$$

which induces a commutative diagram of abelian groups and homomorphisms

$$\begin{array}{ccccc} K(\mathcal{X}, E) & \xleftarrow{j^*} & K(\mathcal{X}, \mathcal{O}) & \xrightarrow{i^*} & K(\mathcal{X}, F) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ K(\mathcal{Y}, E) & \xleftarrow{j^*} & K(\mathcal{Y}, \mathcal{O}) & \xrightarrow{i^*} & K(\mathcal{Y}, F) \end{array}$$

and a similar one for K^\sim . Similar statements hold for $Rf_!$. Therefore, by 6.2, (a) implies (b).

7 Appendix: Compactification and gluing

In [SGA 4, XVII], Deligne studied the gluing of two fibered categories in order to define $Rf_!$ for a compactifiable morphism f of schemes. Two issues prevent us from applying this formalism to Deligne-Mumford stacks: stacks form a 2-category; compactification of coarse spaces only allow us to decompose a morphism into three morphisms (1.10).

To address the first issue, we use the language of pseudo-functors between 2-categories. Since a cofibered category over a category \mathcal{C} can be seen as a pseudo-functor from \mathcal{C} to the 2-category of categories by means of a cleavage, this gives a natural generalization of Deligne's formalism. In fact, we show in 7.5 that the gluing of two pseudo-functors works as in [SGA 4, XVII 3.3.2] without much modification.

To address the second issue, we study the gluing of a finite number of pseudo-functors (7.10). The main result of this appendix is the gluing theorem 7.16, which generalizes 7.5 to more than two pseudo-functors. The key ingredient of the proof is an alternative set of gluing data (7.7, 7.12) which only makes use of 2-Cartesian squares.

At the end of this appendix (7.17 through 7.25), we show that certain maps defined by adjunction satisfy the axioms for gluing data.

We work systematically with 2-categories. For basic notions of 2-categories, see [Bor94, Ch. 7]. As in the main text of this article, for 2-fiber products in a 2-category, we use the convention of [GR09, 1.2.12] for 2-limits, which are called pseudo-bilimits in [Bor94, 7.7].

7.1. Let \mathcal{C} and \mathcal{D} be 2-categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a 2-functor. We say that F is *pseudo-faithful* (resp. *2-faithful*, resp. *2-fully faithful*) if for every pair of objects X and Y of \mathcal{C} , the functor

$$F_{XY}: \mathcal{C}(X, Y) \rightarrow \mathcal{D}(X, Y)$$

is fully faithful (resp. fully faithful and injective on objects, resp. an isomorphism of categories). We say that a 2-subcategory of a 2-category is 2-faithful if the inclusion 2-functor is 2-faithful.

Let \mathcal{D} be a 2-category. A \mathcal{D} -category is a pair (\mathcal{C}, F) consisting of a 2-category \mathcal{C} and a 2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$. If (\mathcal{B}, E) and (\mathcal{C}, F) are \mathcal{D} -categories, a \mathcal{D} -functor $(\mathcal{B}, E) \rightarrow (\mathcal{C}, F)$ is a 2-functor $G: \mathcal{B} \rightarrow \mathcal{C}$

such that $E = FG$. If $G, H: \mathcal{B} \rightarrow \mathcal{C}$ are \mathcal{D} -functors, a \mathcal{D} -natural transformation $\alpha: G \Rightarrow H$ such that $F\alpha: FG \rightarrow FH$ is $\mathbf{1}_E$. A \mathcal{D} -natural equivalence is a \mathcal{D} -natural transformation α that is a 2-natural equivalence. In this case α^{-1} is automatically a \mathcal{D} -natural equivalence. We say that a \mathcal{D} -functor $G: \mathcal{B} \rightarrow \mathcal{C}$ is a \mathcal{D} -equivalence if there exist a \mathcal{D} -functor $H: \mathcal{C} \rightarrow \mathcal{B}$ and \mathcal{D} -natural equivalences $\mathbf{1}_\mathcal{C} \Rightarrow GH$ and $HG \Rightarrow \mathbf{1}_\mathcal{B}$. In this case we say that G and H are \mathcal{D} -quasi-inverse to each other.

A \mathcal{D} -functor $G: \mathcal{B} \rightarrow \mathcal{C}$ is a \mathcal{D} -equivalence if and only if it is 2-fully faithful and every object Y of \mathcal{C} is in the \mathcal{D} -essential image of G , namely, there exists an object X of \mathcal{B} and an invertible 1-cell $GX \rightarrow Y$ of \mathcal{C} whose image in \mathcal{D} is an identity.

7.2. Let C be a set and \mathcal{D} be a 2-category. We view C as a discrete 2-category and denote by $2\text{-Fun}(C, \mathcal{D})$ the 2-category of 2-functors $C \rightarrow \mathcal{D}$. An object of this category is a map $C \rightarrow \text{Ob}(\mathcal{D})$. A 1-cell $\alpha: F \rightarrow F'$ is a family

$$(\alpha(X): FX \rightarrow F'X)_{X \in C}$$

of 1-cells of \mathcal{D} . A 2-cell $\Xi: \alpha \Rightarrow \beta$ is a family

$$(\Xi(X): \alpha(X) \Rightarrow \beta(X))_{X \in C}$$

of 2-cells of \mathcal{D} .

Let \mathcal{C} and \mathcal{D} be 2-categories. We denote by $\text{PsFun}(\mathcal{C}, \mathcal{D})$ the 2-category of pseudo-functors $\mathcal{C} \rightarrow \mathcal{D}$ and view it as a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -category via the forgetful functor

$$|-|: \text{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow 2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$$

The strict fiber 2-categories are 1-categories, that is, 2-categories whose only 2-cells are identities.

A $(2, 1)$ -category is a 2-category whose 2-cells are invertible.

Definition 7.3. Let \mathcal{C} be a $(2, 1)$ -category, \mathcal{A} and \mathcal{B} be two 2-faithful subcategories of \mathcal{C} with $\text{Ob}(\mathcal{A}) = \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{C})$, \mathcal{D} be a 2-category. Define the 2-category $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ of *gluing data* from \mathcal{C} to \mathcal{D} relative to \mathcal{A} and \mathcal{B} as follows. An object of this category is a triple $(F_{\mathcal{A}}, F_{\mathcal{B}}, (G_D))$ consisting of an object $F_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$, an object $F_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ satisfying $|F_{\mathcal{A}}| = |F_{\mathcal{B}}|$, and a family of invertible 2-cells of \mathcal{D}

$$G_D: F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) \Rightarrow F_{\mathcal{B}}(p)F_{\mathcal{A}}(j),$$

D running over squares in \mathcal{C} of the form

$$(7.3.1) \quad \begin{array}{ccc} X & \xrightarrow{j} & Y \\ \downarrow q & \nearrow \alpha & \downarrow p \\ Z & \xrightarrow{i} & W \end{array}$$

where i, j are 1-cells of \mathcal{A} and p, q are 1-cells of \mathcal{B} . The triple is subject to the following conditions:

(a) If D is the square

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ \parallel & \nearrow \alpha & \parallel \\ X & \xrightarrow{i} & Y \end{array}$$

then the following square commutes

$$\begin{array}{ccc} F_{\mathcal{A}}(i) & \xRightarrow{F_{\mathcal{A}}(\alpha)} & F_{\mathcal{A}}(j) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(i)F_{\mathcal{B}}(\mathbf{1}_X) & \xRightarrow{G_D} & F_{\mathcal{B}}(\mathbf{1}_Y)F_{\mathcal{A}}(j) \end{array}$$

(a') If D is the square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow q & \nearrow \alpha & \downarrow p \\ Y & \xlongequal{\quad} & Y \end{array}$$

then the following square commutes

$$\begin{array}{ccc} F_{\mathcal{B}}(q) & \xlongequal{F_{\mathcal{B}}(\alpha)} & F_{\mathcal{B}}(p) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(\mathbf{1}_Y)F_{\mathcal{B}}(q) & \xlongequal{G_D} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(\mathbf{1}_X) \end{array}$$

(b) If D , D' , D'' are respectively the upper, lower and outer squares of the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & Y_1 \\ \downarrow q & \nearrow \alpha & \downarrow p \\ X_2 & \xrightarrow{i_2} & Y_2 \\ \downarrow q' & \nearrow \alpha' & \downarrow p' \\ X_3 & \xrightarrow{i_3} & Y_3 \end{array}$$

then the following pentagon commutes

$$\begin{array}{ccc} F_{\mathcal{A}}(i_3)F_{\mathcal{B}}(q')F_{\mathcal{B}}(q) & \xlongequal{G_{D'}} & F_{\mathcal{B}}(p')F_{\mathcal{A}}(i_2)F_{\mathcal{B}}(q) \xlongequal{G_D} F_{\mathcal{B}}(p')F_{\mathcal{B}}(p)F_{\mathcal{A}}(i_1) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(i_3)F_{\mathcal{B}}(q'q) & \xlongequal{G_{D''}} & F_{\mathcal{B}}(p'p)F_{\mathcal{A}}(i_1) \end{array}$$

(b') If D , D' , D'' are respectively the left, right and outer squares of the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\ \downarrow p_1 & \nearrow \alpha & \downarrow p_2 & \nearrow \alpha' & \downarrow p_3 \\ Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3 \end{array}$$

then the following pentagon commutes

$$\begin{array}{ccc} F_{\mathcal{A}}(i')F_{\mathcal{A}}(i)F_{\mathcal{B}}(p_1) & \xlongequal{G_D} & F_{\mathcal{A}}(i')F_{\mathcal{B}}(p_2)F_{\mathcal{A}}(j) \xlongequal{G_{D'}} F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(j')F_{\mathcal{A}}(j) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(i'i)F_{\mathcal{B}}(p) & \xlongequal{G_{D''}} & F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(j'j) \end{array}$$

A 1-cell $(F_{\mathcal{A}}, F_{\mathcal{B}}, G) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}})$ consisting of a 1-cell $\alpha_{\mathcal{A}}: F_{\mathcal{A}} \rightarrow F'_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a 1-cell $\alpha_{\mathcal{B}}: F_{\mathcal{B}} \rightarrow F'_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|$ and that for any square D (7.3.1), the following hexagon commutes

$$(7.3.2) \quad \begin{array}{ccccc} \alpha_0(W)F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xlongequal{\alpha_{\mathcal{A}}(i)} & F'_{\mathcal{A}}(i)\alpha_0(Z)F_{\mathcal{B}}(q) & \xlongequal{\alpha_{\mathcal{B}}(q)} & F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(q)\alpha_0(X) \\ \Downarrow G_D & & & & \Downarrow G'_D \\ \alpha_0(W)F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xlongequal{\alpha_{\mathcal{B}}(p)} & F'_{\mathcal{B}}(p)\alpha_0(Y)F_{\mathcal{A}}(j) & \xlongequal{\alpha_{\mathcal{A}}(j)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(j)\alpha_0(X) \end{array}$$

where $\alpha_0 = |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|$.

A 2-cell of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}): (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \Rightarrow (\alpha'_{\mathcal{A}}, \alpha'_{\mathcal{B}})$ consisting of a 2-cell $\Xi_{\mathcal{A}}: \alpha_{\mathcal{A}} \Rightarrow \alpha'_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a 2-cell $\Xi_{\mathcal{B}}: \alpha_{\mathcal{B}} \Rightarrow \alpha'_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|$.

We view $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$ as a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -category via the functor given by

$$\begin{aligned} (F_{\mathcal{A}}, F_{\mathcal{B}}, G) &\mapsto |F_{\mathcal{A}}| = |F_{\mathcal{B}}|, \\ (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) &\mapsto |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|, \quad (\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}) \mapsto |\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|. \end{aligned}$$

Let $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ be an object of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$. For every square D (7.3.1), let $G_{D'}^* = G_D^{-1}$, where D' is the square obtained from D by inverting α . Then $(F_{\mathcal{B}}, F_{\mathcal{A}}, G^*)$ is an object of $\text{GD}_{\mathcal{B},\mathcal{A}}(\mathcal{C}, \mathcal{D})$. If $(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ is a 1-cell of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$, then

$$(\alpha_{\mathcal{B}}, \alpha_{\mathcal{A}}): (F_{\mathcal{B}}, F_{\mathcal{A}}, G^*) \rightarrow (F'_{\mathcal{B}}, F'_{\mathcal{A}}, G'^*)$$

is a 1-cell of $\text{GD}_{\mathcal{B},\mathcal{A}}(\mathcal{C}, \mathcal{D})$. If $(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}): (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \Rightarrow (\alpha'_{\mathcal{A}}, \alpha'_{\mathcal{B}})$ is a 2-cell of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$, then

$$(\Xi_{\mathcal{B}}, \Xi_{\mathcal{A}}): (\alpha_{\mathcal{B}}, \alpha_{\mathcal{A}}) \Rightarrow (\alpha'_{\mathcal{B}}, \alpha'_{\mathcal{A}})$$

is a 2-cell of $\text{GD}_{\mathcal{B},\mathcal{A}}(\mathcal{C}, \mathcal{D})$. This defines an isomorphism of $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -categories

$$\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \text{GD}_{\mathcal{B},\mathcal{A}}(\mathcal{C}, \mathcal{D}).$$

Let F be an object of $\text{PsFun}(\mathcal{C}, \mathcal{D})$. For every square D (7.3.1), define G_D to be the composition

$$F(i)F(q) \Longrightarrow F(iq) \xrightarrow{F(\alpha)} F(pj) \Longrightarrow F(p)F(j).$$

Then $(F|_{\mathcal{A}}, F|_{\mathcal{B}}, G)$ is an object of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$. If $\alpha: F \rightarrow F'$ is a 1-cell of $\text{PsFun}(\mathcal{C}, \mathcal{D})$, then

$$(\alpha|_{\mathcal{A}}, \alpha|_{\mathcal{B}}): (F|_{\mathcal{A}}, F|_{\mathcal{B}}, G) \rightarrow (F'|_{\mathcal{A}}, F'|_{\mathcal{B}}, G')$$

is a 1-cell of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$. If $\Xi: \alpha \Rightarrow \alpha'$ is a 2-cell of $\text{PsFun}(\mathcal{A}, \mathcal{B})$, then

$$(\Xi|_{\mathcal{A}}, \Xi|_{\mathcal{B}}): (\alpha|_{\mathcal{A}}, \alpha|_{\mathcal{B}}) \Rightarrow (\alpha'|_{\mathcal{A}}, \alpha'|_{\mathcal{B}})$$

is a 2-cell of $\text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$. This defines a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -functor

$$(7.3.3) \quad \text{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D}).$$

7.4. We say that \mathcal{A} and \mathcal{B} *generate* \mathcal{C} if every 1-cell of \mathcal{C} is isomorphic to a composition $p_1 i_1 \dots p_n i_n$, where i_1, \dots, i_n are 1-cells of \mathcal{A} and p_1, \dots, p_n are 1-cells of \mathcal{B} . In this case, (7.3.3) is 2-faithful since the restriction 2-functor

$$\text{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{PsFun}(\mathcal{A}, \mathcal{D}) \times_{2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})} \text{PsFun}(\mathcal{B}, \mathcal{D})$$

is 2-faithful. Here the fiber product is strict. To see the 2-faithfulness, let $\alpha: F \rightarrow F'$ be a 1-cell of $\text{PsFun}(\mathcal{C}, \mathcal{D})$. If $f: X \rightarrow Y$ is a 1-cell of \mathcal{C} and

$$\gamma: p_1 i_1 \dots p_n i_n \Rightarrow f$$

is a 2-cell of \mathcal{C} , where i_1, \dots, i_n are 1-cells of \mathcal{A} and p_1, \dots, p_n are 1-cells of \mathcal{B} . then the following diagram commutes

$$\begin{array}{ccccc} \alpha(Y)F(p_1)F(i_1) \dots F(p_n)F(i_n) & \Longrightarrow & \alpha(Y)F(p_1 i_1 \dots p_n i_n) & \xrightarrow{F(\gamma)} & \alpha(Y)F(f) \\ \Downarrow \alpha(p_1)\alpha(i_1) \dots \alpha(p_n)\alpha(i_n) & & \Downarrow \alpha(p_1 i_1 \dots p_n i_n) & & \Downarrow \alpha(f) \\ F'(p_1)F'(i_1) \dots F'(p_n)F'(i_n)\alpha(X) & \Longrightarrow & F'(p_1 i_1 \dots p_n i_n)\alpha(X) & \xrightarrow{F'(\gamma)} & F'(f)\alpha(X) \end{array}$$

Hence α is determined by $\alpha|_{\mathcal{A}}$ and $\alpha|_{\mathcal{B}}$. Let $\alpha, \beta: F \rightarrow F'$ be 1-cells of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ and $\Xi: \alpha \Rightarrow \beta$ be a 2-cell of $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$. For any triangle in \mathcal{C}

$$\begin{array}{ccc} X & & \\ \downarrow f & \nearrow \gamma & \searrow h \\ Y & \xrightarrow{g} & Z \end{array}$$

the leftmost and the rightmost inner squares of the following diagram commute

$$\begin{array}{ccccccc} & & \alpha(h) & & & & \\ & \xRightarrow{\quad} & \xRightarrow{\quad} & \xRightarrow{\quad} & \xRightarrow{\quad} & \xRightarrow{\quad} & \\ \alpha(Z)F(h) & \xRightarrow{F(\gamma^{-1})} & \alpha(Z)F(g)F(f) & \xRightarrow{\alpha(g)} & F'(g)\alpha(Y)F(f) & \xRightarrow{\alpha(f)} & F'(g)F'(f)\alpha(X) & \xRightarrow{F'(\gamma)} & F'(h)\alpha(X) \\ \downarrow \Xi_Z & & \downarrow \Xi_Z & & \downarrow \Xi_Y & & \downarrow \Xi_X & & \downarrow \Xi_X \\ \beta(Z)F(h) & \xRightarrow{F(\gamma^{-1})} & \beta(Z)F(g)F(f) & \xRightarrow{\beta(g)} & F'(g)\beta(Y)F(f) & \xRightarrow{\beta(f)} & F'(g)F'(f)\beta(X) & \xRightarrow{F'(\gamma)} & F'(h)\beta(X) \\ & & \beta(h) & & & & & & \end{array}$$

Hence Ξ is a 2-cell of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ if and only if it is both a 2-cell of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a 2-cell of $\text{PsFun}(\mathcal{B}, \mathcal{D})$.

The following is a generalization of [SGA 4, XVII 3.3.2] to 2-categories.

Proposition 7.5. *Let \mathcal{C} be a $(2, 1)$ -category, \mathcal{A} and \mathcal{B} be two 2-faithful subcategories (7.1) of \mathcal{C} with $\text{Ob}(\mathcal{A}) = \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{C})$, \mathcal{D} be a 2-category. Assume the following:*

- (i) *Every 1-cell of \mathcal{C} is isomorphic to pi for some 1-cell i of \mathcal{A} and some 1-cell p of \mathcal{B} .*
- (ii) *2-fiber products exist in \mathcal{B} and are 2-fiber products in \mathcal{C} .*

Then (7.3.3) is a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -equivalence (7.1).

Proof. The 2-functor is 2-faithful since (i) implies that \mathcal{A} and \mathcal{B} generate \mathcal{C} .

For any 1-cell f of \mathcal{C} , consider the 2-category of *compactifications* of f . An object of this 2-category is a quadruple (Z, i, p, α) consisting of an object Z of \mathcal{C} , a 1-cell $i: X \rightarrow Z$ of \mathcal{A} , a 1-cell $p: Z \rightarrow Y$ of \mathcal{B} , and a 2-cell $\alpha: f \Rightarrow pi$ of \mathcal{C} corresponding to the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ f \downarrow \alpha & \nearrow p & \\ Y & & \end{array}$$

A 1-cell $(Z, i, p, \alpha) \rightarrow (W, j, q, \beta)$ is a triple (r, γ, δ) consisting of a 1-cell $r: Z \rightarrow W$ of \mathcal{B} , and 2-cells $\gamma: ri \Rightarrow j$ and $\delta: p \Rightarrow qr$ of \mathcal{C} , fitting in the diagram

$$\begin{array}{ccccc} & X & & & \\ & \downarrow i & \nearrow j & & \\ f \curvearrowright \alpha & Z & \xrightarrow{r} & W & \\ & \downarrow p & \searrow q & & \\ & Y & & & \end{array}$$

where the outer triangle is β . A 2-cell $(r, \gamma, \delta) \Rightarrow (r', \gamma', \delta')$ is a 2-cell $\epsilon: r \Rightarrow r'$ of \mathcal{B} such that $\gamma = \gamma'(\epsilon i)$ and $\delta' = \delta(q\epsilon)$. We denote by $\text{Comp}(f)$ the 1-category obtained by identifying isomorphic 1-cells. Under the assumptions of 7.5, one shows as in [SGA 4, XVII 3.2.6] that $\text{Comp}(f)^{\text{op}}$ is a filtrant category.

Let F and F' be objects of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ and

$$(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}): (F|_{\mathcal{A}}, F|_{\mathcal{B}}, G) \rightarrow (F'|_{\mathcal{A}}, F'|_{\mathcal{B}}, G')$$

be a 1-cell of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ whose source and target are respectively the images of F and F' under (7.3.3). Define a 1-cell $\epsilon: F \rightarrow F'$ of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ as follows. For any object X of \mathcal{C} , let $\epsilon(X) = \epsilon_{\mathcal{A}}(X) = \epsilon_{\mathcal{B}}(X)$. For any 1-cell $f: X \rightarrow Y$ of \mathcal{C} , let $\epsilon(f)$ be the unique 2-cell in \mathcal{D} making the following diagram commute

$$\begin{array}{ccccc} \epsilon(Y)F(p)F(i) & \xRightarrow{\quad} & \epsilon(Y)F(pi) & \xRightarrow{F(\alpha^{-1})} & \epsilon(Y)F(f) \\ \downarrow \epsilon_{\mathcal{B}}(p)\epsilon_{\mathcal{A}}(i) & & & & \downarrow \epsilon(f) \\ F'(p)F'(i)\epsilon(X) & \xRightarrow{\quad} & F'(pi)\epsilon(X) & \xRightarrow{F'(\alpha^{-1})} & F'(f)\epsilon(X) \end{array}$$

where (Z, i, p, α) is a compactification of f . This does not depend on the choice of the compactification, because, if $(r, \gamma, \delta): (Z, i, p, \alpha) \rightarrow (W, j, q, \beta)$ is a 1-cell of compactifications, then the following diagram commutes

$$\begin{array}{ccccccc} & & & F(\beta) & & & \\ & & & \curvearrowright & & & \\ \epsilon(Y)F(f) & \xRightarrow{F(\alpha)} & \epsilon(Y)F(p)F(i) & \xRightarrow{F(\delta)} & \epsilon(Y)F(q)F(r)F(i) & \xRightarrow{G_D^{-1}} & \epsilon(Y)F(q)F(j) \\ & & \downarrow \epsilon_{\mathcal{B}}(p)\epsilon_{\mathcal{A}}(i) & & \downarrow \epsilon_{\mathcal{B}}(q)\epsilon_{\mathcal{B}}(r)\epsilon_{\mathcal{A}}(i) & & \downarrow \epsilon_{\mathcal{B}}(q)\epsilon_{\mathcal{A}}(j) \\ F'(f)\epsilon(X) & \xRightarrow{F'(\alpha)} & F'(p)F'(i)\epsilon(X) & \xRightarrow{F'(\delta)} & F'(q)F'(r)F'(i)\epsilon(X) & \xRightarrow{G_D'^{-1}} & F'(q)F'(j)\epsilon(X) \\ & & & \curvearrowright & & & \\ & & & F'(\beta) & & & \end{array}$$

where D is the square

$$(7.5.1) \quad \begin{array}{ccc} X & \xrightarrow{i} & Z \\ \parallel & \nearrow \gamma^{-1} & \downarrow r \\ X & \xrightarrow{j} & W \end{array}$$

For any object X of \mathcal{C} , $(X, \mathbf{1}_X, \mathbf{1}_X, \mathbf{1}_X)$ is a compactification of X , so the following diagram commutes

$$\begin{array}{ccccc} \epsilon(X) & \xRightarrow{\quad} & \epsilon(X)F(\mathbf{1}_X)F(\mathbf{1}_X) & \xRightarrow{\quad} & \epsilon(X)F(\mathbf{1}_X) \\ & \searrow & \downarrow \epsilon_{\mathcal{A}}(\mathbf{1}_X)\epsilon_{\mathcal{B}}(\mathbf{1}_X) & & \downarrow \epsilon(\mathbf{1}_X) \\ & & F'(\mathbf{1}_X)F'(\mathbf{1}_X)\epsilon(X) & \xRightarrow{\quad} & F'(\mathbf{1}_X)\epsilon(X) \end{array}$$

For any composable pair of 1-cells $X \xrightarrow{f} Y \xrightarrow{g} Z$, choose a diagram

$$(7.5.2) \quad \begin{array}{ccccc} X & \xrightarrow{i} & X' & \xrightarrow{k} & X'' \\ f \downarrow \alpha & \nearrow p & \nearrow \gamma & \nearrow r & \\ Y & \xrightarrow{j} & Y' & & \\ g \downarrow \beta & \nearrow q & & & \\ Z & & & & \end{array}$$

such that i, j, k are 1-cells of \mathcal{A} and p, q, r are 1-cells of \mathcal{B} . If we denote by D the square containing γ

and by $\delta: qrki \Rightarrow gf$ the 2-cell represented by the outer triangle, then the following diagram commutes

$$\begin{array}{ccccccc}
\epsilon(Z)F(g)F(f) & \xRightarrow{F(\beta)F(\alpha)} & \epsilon(Z)F(q)F(j)F(p)F(i) & \xRightarrow{G_D} & \epsilon(Z)F(q)F(r)F(k)F(i) & \xleftarrow{F(\delta)} & \epsilon(Z)F(gf) \\
\Downarrow \epsilon(f)\epsilon(g) & & \Downarrow \epsilon_{\mathcal{B}}(q)\epsilon_{\mathcal{A}}(j)\epsilon_{\mathcal{B}}(p)\epsilon_{\mathcal{A}}(i) & & \Downarrow \epsilon_{\mathcal{B}}(q)\epsilon_{\mathcal{B}}(r)\epsilon_{\mathcal{A}}(k)\epsilon_{\mathcal{A}}(i) & & \Downarrow \epsilon(gf) \\
F'(g)F'(f)\epsilon(X) & \xRightarrow{F'(\beta)F'(\alpha)} & F'(q)F'(j)F'(p)F'(i)\epsilon(X) & \xRightarrow{G'_D} & F'(q)F'(r)F'(k)F'(i)\epsilon(X) & \xleftarrow{F'(\delta)} & F'(gf)\epsilon(X)
\end{array}$$

Hence ϵ is a pseudo-natural transformation. Its image under (7.3.3) is clearly $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$. Therefore (7.3.3) is 2-fully faithful.

It remains to show that every gluing datum $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ is in the $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -essential image of (7.3.3). The proof is similar to that of [SGA 4, XVII 3.3.2]. We give a brief sketch. We construct an object F of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ as follows. For any object X of \mathcal{C} , let $FX = F_{\mathcal{A}}X = F_{\mathcal{B}}X$. For any 1-cell $f: X \rightarrow Y$ of \mathcal{C} , consider the functor $F_f: \text{Comp}(f) \rightarrow \mathcal{D}(FX, FY)$ which, to a compactification (Z, i, p, α) of f , associates the 1-cell $F_{\mathcal{B}}(p)F_{\mathcal{A}}(i): FX \rightarrow FY$ of \mathcal{D} , and, to a morphism of compactifications $(r, \gamma, \delta): (Z, i, p, \alpha) \rightarrow (W, j, q, \beta)$, associate the invertible 2-cell

$$F_{\mathcal{B}}(p)F_{\mathcal{A}}(i) \xRightarrow{F_{\mathcal{B}}(\delta)} F_{\mathcal{B}}(q)F_{\mathcal{B}}(r)F_{\mathcal{A}}(i) \xRightarrow{G_D^{-1}} F_{\mathcal{B}}(q)F_{\mathcal{A}}(j),$$

where D is the square (7.5.1). Since $\text{Comp}(f)^{\text{op}}$ is filtrant, F_f defines, for every pair of compactifications $(Z, i, p, \alpha), (W, j, q, \beta)$ of f , an invertible 2-cell

$$F_{\mathcal{B}}(p)F_{\mathcal{A}}(i) \Rightarrow F_{\mathcal{B}}(q)F_{\mathcal{A}}(j).$$

Choose one compactification (Z, i, p, α) of f and let $Ff = F_{\mathcal{B}}(p)F_{\mathcal{A}}(i)$. For any 2-cell

$$\begin{array}{ccc}
& f & \\
X & \Downarrow \phi & Y \\
& g &
\end{array}$$

in \mathcal{C} , choose a decomposition of ϕ

$$\begin{array}{ccccc}
& & X & & \\
& \nearrow i & \downarrow k & \nwarrow j & \\
& & X' & & \\
& \searrow r & \downarrow \epsilon & \swarrow s & \\
f & \Rightarrow \alpha & Z & \xRightarrow{\epsilon} & W & \Rightarrow \beta^{-1} & g \\
& \searrow p & & \swarrow q & & \\
& & Y & &
\end{array}$$

where (Z, i, p, α) and (W, j, q, β) are the chosen compactifications of f and g , respectively, k is a 1-cell of \mathcal{A} and r, s are 1-cells of \mathcal{B} . Let $F\phi: Ff \Rightarrow Fg$ be the composition

$$F_{\mathcal{B}}(p)F_{\mathcal{A}}(i) \xRightarrow{G_D} F_{\mathcal{B}}(p)F_{\mathcal{B}}(r)F_{\mathcal{A}}(k) \xRightarrow{F_{\mathcal{B}}(\epsilon)} F_{\mathcal{B}}(q)F_{\mathcal{B}}(s)F_{\mathcal{A}}(k) \xRightarrow{G_{D'}^{-1}} F_{\mathcal{B}}(q)F_{\mathcal{A}}(j),$$

where D and D' are respectively the squares

$$\begin{array}{ccc}
X & \xrightarrow{k} & X' \\
\parallel & \nearrow \gamma & \downarrow r \\
X & \xrightarrow{i} & Z
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{k} & X' \\
\parallel & \nearrow \delta & \downarrow s \\
X & \xrightarrow{j} & W
\end{array}$$

This does not depend on the choice of the decomposition of ϕ . For any object X of \mathcal{C} , $(X, \mathbf{1}_X, \mathbf{1}_X, \mathbf{1}_{\mathbf{1}_X})$ is a compactification of $\mathbf{1}_X$ and induces an invertible 2-cell

$$\mathbf{1}_{FX} \Longrightarrow F_{\mathcal{B}}(\mathbf{1}_X)F_{\mathcal{A}}(\mathbf{1}_X) \xRightarrow{F_{\mathbf{1}_X}} F(\mathbf{1}_X) .$$

For any composable pair of 1-cells $X \xrightarrow{f} Y \xrightarrow{g} Z$, choose a diagram (7.5.2) where (Z, i, p, α) and (W, j, q, β) are the chosen compactifications of f and g , respectively. Associate the following invertible 2-cell $F(g)F(f) \Rightarrow F(gf)$:

$$F_{\mathcal{B}}(q)F_{\mathcal{A}}(j)F_{\mathcal{B}}(p)F_{\mathcal{A}}(i) \xRightarrow{G_D} F_{\mathcal{B}}(q)F_{\mathcal{B}}(r)F_{\mathcal{A}}(k)F_{\mathcal{A}}(i) \Longrightarrow F_{\mathcal{B}}(qr)F_{\mathcal{A}}(ki) \xRightarrow{F_{gf}} F(gf) ,$$

where D is the square in (7.5.2) containing γ . One verifies that this does not depend on the choice of (7.5.2) and F is a pseudo-functor. Define invertible 1-cells $\epsilon_{\mathcal{A}}: F_{\mathcal{A}} \rightarrow F|_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and $\epsilon_{\mathcal{B}}: F_{\mathcal{B}} \rightarrow F|_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ as follows. For any object X of \mathcal{C} , let $\epsilon_{\mathcal{A}}(X) = \epsilon_{\mathcal{B}}(X) = \mathbf{1}_{FX}$. For any 1-cell $i: X \rightarrow Y$ of \mathcal{A} , $(Y, i, \mathbf{1}_Y, \mathbf{1}_i)$ is a compactification of i . Let $\epsilon_{\mathcal{A}}(i): F_{\mathcal{A}}(i) \Rightarrow F(i)$ be the 2-cell induced by F_i . For any 1-cell $p: X \rightarrow Y$ of \mathcal{B} , $(X, \mathbf{1}_X, p, \mathbf{1}_p)$ is a compactification of p . Let $\epsilon_{\mathcal{B}}(p): F_{\mathcal{B}}(p) \Rightarrow F(p)$ be the 2-cell induced by F_p . Then $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ gives an isomorphism from $(F_{\mathcal{A}}, F_{\mathcal{B}}, G_D)$ to the image of F under (7.3.3). \square

7.6. Let $\mathcal{C}, \mathcal{A}, \mathcal{B}, \mathcal{D}$ be as in 7.3. Let $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ be an object of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, $F_0 = |F_{\mathcal{A}}| = |F_{\mathcal{B}}|$. Define an invertible 1-cell $\rho: F_{\mathcal{B}}|_{\mathcal{A} \cap \mathcal{B}} \rightarrow F_{\mathcal{A}}|_{\mathcal{A} \cap \mathcal{B}}$ of $\text{PsFun}(\mathcal{A} \cap \mathcal{B}, \mathcal{D})$ with $|\rho| = \mathbf{1}_{F_0}$ as follows. For any 1-cell $f: X \rightarrow Y$ of $\mathcal{A} \cap \mathcal{B}$, let $\rho(f)$ be the composition

$$F_{\mathcal{B}}(f) \Longrightarrow F_{\mathcal{A}}(\mathbf{1}_Y)F_{\mathcal{B}}(f) \xRightarrow{G_D} F_{\mathcal{B}}(\mathbf{1}_Y)F_{\mathcal{A}}(f) \Longrightarrow F_{\mathcal{A}}(f) ,$$

where D is left square in the diagram

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \xrightarrow{f} & Y \\ \parallel & & \downarrow f & & \parallel \\ X & \xrightarrow{f} & Y & \xlongequal{\quad} & Y \end{array}$$

Denote the right square by D' . Applying axiom (a) in 7.3 to the outer square and axiom (b') to the above diagram, one sees that $\rho(f)$ is the inverse of the composition

$$F_{\mathcal{A}}(f) \Longrightarrow F_{\mathcal{A}}(f)F_{\mathcal{B}}(\mathbf{1}_X) \xRightarrow{G_{D'}} F_{\mathcal{B}}(f)F_{\mathcal{A}}(\mathbf{1}_X) \Longrightarrow F_{\mathcal{B}}(f) .$$

For any object X of \mathcal{C} , applying axiom (a) to the commutative square consisting of X and $\mathbf{1}_X$, one finds that the following diagram commutes

$$\begin{array}{ccc} \mathbf{1}_{F_0 X} & & \\ \parallel & \searrow & \\ F_{\mathcal{B}}(\mathbf{1}_X) & \xRightarrow{\rho(\mathbf{1}_X)} & F_{\mathcal{A}}(\mathbf{1}_X) \end{array}$$

For any composable pair of 1-cells $X \xrightarrow{f} Y \xrightarrow{g} Z$, applying axioms (a), (b), (b') to

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow f & \parallel & \parallel & \parallel & \parallel \\ Y & \xlongequal{\quad} & Y & \xrightarrow{g} & Z \\ \downarrow g & & \parallel & & \parallel \\ Z & \xrightarrow{\quad} & Z & & \end{array}$$

one finds that the following diagram commutes

$$\begin{array}{ccc} F_{\mathcal{B}}(g)F_{\mathcal{B}}(f) & \xRightarrow{\rho(g)\rho(f)} & F_{\mathcal{A}}(g)F_{\mathcal{A}}(f) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{B}}(gf) & \xRightarrow{\rho(gf)} & F_{\mathcal{A}}(gf) \end{array}$$

Hence ρ is a pseudo-natural equivalence.

We claim that ρ has the following properties:

(c) If D is a square (7.3.1) such that p, q are 1-cells of $\mathcal{A} \cap \mathcal{B}$, then the following hexagon commutes

$$\begin{array}{ccccc} F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xRightarrow{\rho(q)} & F_{\mathcal{A}}(i)F_{\mathcal{A}}(q) & \Longrightarrow & F_{\mathcal{A}}(iq) \\ \Downarrow G_D & & & & \Downarrow F_{\mathcal{A}}(\alpha) \\ F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xRightarrow{\rho(p)} & F_{\mathcal{A}}(p)F_{\mathcal{A}}(j) & \Longrightarrow & F_{\mathcal{A}}(pj) \end{array}$$

(c') If D is a square (7.3.1) such that i, j are 1-cells of $\mathcal{A} \cap \mathcal{B}$, then the following hexagon commutes

$$\begin{array}{ccccc} F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xRightarrow{\rho(i)^{-1}} & F_{\mathcal{B}}(i)F_{\mathcal{B}}(q) & \Longrightarrow & F_{\mathcal{B}}(iq) \\ \Downarrow G_D & & & & \Downarrow F_{\mathcal{B}}(\alpha) \\ F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xRightarrow{\rho(j)^{-1}} & F_{\mathcal{B}}(p)F_{\mathcal{B}}(j) & \Longrightarrow & F_{\mathcal{B}}(pj) \end{array}$$

In fact, any square D in (c) can be decomposed as

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \xlongequal{\quad} & Y \\ \Downarrow & \swarrow & \Downarrow & \swarrow & \downarrow p \\ X & \xrightarrow{j} & Y & \xrightarrow{p} & W \\ \Downarrow & & \alpha \nearrow & & \Downarrow \\ X & \xrightarrow{q} & Z & \xrightarrow{i} & W \\ \downarrow q & \swarrow & \Downarrow & \swarrow & \Downarrow \\ Z & \xlongequal{\quad} & Z & \xrightarrow{i} & W \end{array}$$

Denote the upper left, upper right, middle, lower left, lower right squares by D_1, D_2, D_3, D_4 and D_5 , respectively. Then G_{D_2} can be identified with $\rho(p)^{-1}$ and G_{D_4} can be identified with $\rho(q)$. By axiom (a), G_{D_1} and G_{D_5} can be identified with identities and G_{D_3} can be identified with $F_{\mathcal{A}}(\alpha)$. Hence axioms (b) and (b') imply that the hexagon in (c) commutes. Similarly, axioms (a'), (b) and (b') imply (c').

Definition 7.7. Define a category $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ as follows. An object of this category is a quadruple $(F_{\mathcal{A}}, F_{\mathcal{B}}, (G_D), \rho)$ consisting of an object $F_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$, an object $F_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$, a family of invertible 2-cells of \mathcal{D}

$$G_D: F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) \Rightarrow F_{\mathcal{B}}(p)F_{\mathcal{A}}(j),$$

D running over 2-Cartesian squares in \mathcal{C} of the form (7.3.1), and an invertible 1-cell $\rho: F_{\mathcal{B}}|_{\mathcal{A} \cap \mathcal{B}} \rightarrow F_{\mathcal{A}}|_{\mathcal{A} \cap \mathcal{B}}$ of $\text{PsFun}(\mathcal{A} \cap \mathcal{B}, \mathcal{D})$, such that $|\rho| = \mathbf{1}_{F_0}$, where $F_0 = |F_{\mathcal{A}}| = |F_{\mathcal{B}}|$, and satisfying conditions (b), (b') in 7.3 and conditions (c), (c') above for 2-Cartesian squares.

A 1-cell $(F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G', \rho')$ of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}})$ consisting of a 1-cell $\alpha_{\mathcal{A}}: F_{\mathcal{A}} \rightarrow F'_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a 1-cell $\alpha_{\mathcal{B}}: F_{\mathcal{B}} \rightarrow F'_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$, such that $|\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|$, that for any 2-Cartesian square D (7.3.1), the diagram (7.3.2) commutes, and that the following square commutes

$$\begin{array}{ccc} F_{\mathcal{B}}|_{\mathcal{A} \cap \mathcal{B}} & \xrightarrow{\rho} & F_{\mathcal{A}}|_{\mathcal{A} \cap \mathcal{B}} \\ \downarrow \alpha_{\mathcal{B}}|_{\mathcal{A} \cap \mathcal{B}} & & \downarrow \alpha_{\mathcal{A}}|_{\mathcal{A} \cap \mathcal{B}} \\ F'_{\mathcal{B}}|_{\mathcal{A} \cap \mathcal{B}} & \xrightarrow{\rho'} & F'_{\mathcal{A}}|_{\mathcal{A} \cap \mathcal{B}} \end{array}$$

A 2-cell of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}): (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \Rightarrow (\alpha'_{\mathcal{A}}, \alpha'_{\mathcal{B}})$ consisting of a 2-cell $\Xi_{\mathcal{A}}: \alpha_{\mathcal{A}} \Rightarrow \alpha'_{\mathcal{A}}$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a 2-cell $\Xi_{\mathcal{B}}: \alpha_{\mathcal{B}} \Rightarrow \alpha'_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|$.

We view $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ as a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -category via the functor given by

$$\begin{aligned} (F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho) &\mapsto |F_{\mathcal{A}}| = |F_{\mathcal{B}}|, \\ (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) &\mapsto |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|, \quad (\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}) \mapsto |\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|. \end{aligned}$$

The construction 7.6 defines a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -functor

$$(7.7.1) \quad \text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D}),$$

which is clearly 2-faithful.

7.8. We say that \mathcal{A} and \mathcal{B} are *squaring* in \mathcal{C} if every square (7.3.1) can be decomposed as

$$(7.8.1) \quad \begin{array}{ccccc} X & & & & \\ & \searrow f & & \nearrow j & \\ & & X' & \xrightarrow{k} & Y \\ & \nearrow \gamma & \downarrow r & \searrow \beta & \downarrow p \\ & & Z & \xrightarrow{i} & W \end{array}$$

where k is a 1-cell of \mathcal{A} , r is a 1-cell of \mathcal{B} , f is a 1-cell of $\mathcal{A} \cap \mathcal{B}$, and the inner square is 2-Cartesian in \mathcal{C} . If $\mathcal{A} = \mathcal{B}$, a sufficient condition for \mathcal{A} and \mathcal{A} to be squaring in \mathcal{C} is that 2-fiber products exist in \mathcal{A} and are 2-fiber products in \mathcal{C} . In general, a sufficient condition for \mathcal{A} and \mathcal{B} to be squaring is that 2-fiber products exist in \mathcal{C} , and \mathcal{A} and \mathcal{B} are closed under 2-base change in \mathcal{C} and taking the diagonal in \mathcal{C} .

If \mathcal{A} and \mathcal{B} are squaring, then (7.7.1) is 2-fully faithful. In fact, for objects $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$, $(F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ and any morphism

$$(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G', \rho')$$

of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ whose source and target are respectively the images of $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ and $(F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ under (7.7.1), $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ is a morphism of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. Indeed, for any square D (7.3.1), decomposing it as (7.8.1), we see that the following diagram commutes

$$\begin{array}{ccccccc} & & & G_D & & & \\ \epsilon_0(W)F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xRightarrow{F_{\mathcal{B}}(\gamma)} & \epsilon_0(W)F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) & \xRightarrow{G_{D'}\rho(f)} & \epsilon_0(W)F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f) & \xRightarrow{F_{\mathcal{A}}(\delta)} & \epsilon_0(W)F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) \\ \downarrow \epsilon_{\mathcal{A}}(i)\epsilon_{\mathcal{B}}(q) & & \downarrow \epsilon_{\mathcal{A}}(i)\epsilon_{\mathcal{B}}(r)\epsilon_{\mathcal{B}}(f) & & \downarrow \epsilon_{\mathcal{B}}(p)\epsilon_{\mathcal{A}}(k)\epsilon_{\mathcal{A}}(f) & & \downarrow \epsilon_{\mathcal{B}}(p)\epsilon_{\mathcal{A}}(j) \\ F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(q)\epsilon_0(X) & \xRightarrow{F'_{\mathcal{B}}(\gamma)} & F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(r)F'_{\mathcal{B}}(f)\epsilon_0(X) & \xRightarrow{G'_{D'}\rho'(f)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(k)F'_{\mathcal{A}}(f)\epsilon_0(X) & \xRightarrow{F'_{\mathcal{A}}(\delta)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(j)\epsilon_0(X) \\ & & & G'_D & & & \end{array}$$

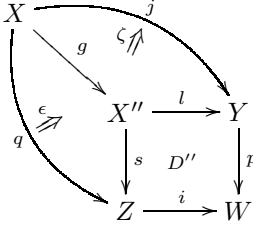
where $\epsilon_0 = |\epsilon_{\mathcal{A}}| = |\epsilon_{\mathcal{B}}|$.

Proposition 7.9. *Let \mathcal{C} be a $(2, 1)$ -category, \mathcal{A} and \mathcal{B} be two 2-faithful subcategories of \mathcal{C} with $\text{Ob}(\mathcal{A}) = \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{C})$, \mathcal{D} be a 2-category. Assume that every 1-cell of \mathcal{C} that is an equivalence is contained in $\mathcal{A} \cap \mathcal{B}$, \mathcal{A} and \mathcal{B} are squaring in \mathcal{C} , \mathcal{A} and $\mathcal{A} \cap \mathcal{B}$ are squaring in \mathcal{C} , and \mathcal{B} and $\mathcal{A} \cap \mathcal{B}$ are squaring in \mathcal{C} . Then (7.7.1) is an isomorphism of 2-categories.*

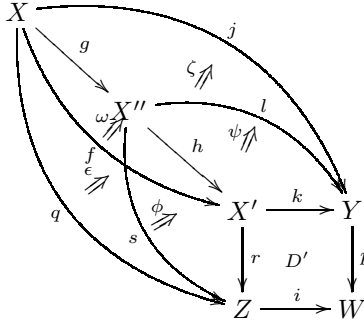
Proof. We construct the inverse as follows. Let $(F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho)$ be an object of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. For any square D (7.3.1), decompose it as (7.8.1), and denote the inner square by D' . Let \bar{G}_D be the composition

$$F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) \xrightarrow{F_{\mathcal{B}}(\gamma)} F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) \xrightarrow{G_{D'}\rho(f)} F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f) \xrightarrow{F_{\mathcal{A}}(\delta)} F_{\mathcal{B}}(p)F_{\mathcal{A}}(j)$$

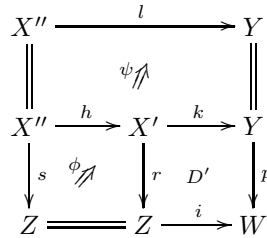
This does not depend on the choice of the decomposition. In fact, if



is another decomposition with l in \mathcal{A} , s in \mathcal{B} , g in $\mathcal{A} \cap \mathcal{B}$, D'' 2-Cartesian in \mathcal{C} , then they can be combined into



where h is an equivalence. Applying the axioms to the decomposition of D''



we obtain the following commutative diagram

$$\begin{array}{ccc} F_{\mathcal{A}}(i)F_{\mathcal{B}}(s) & \xrightarrow{\quad G_{D''} \quad} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(l) \\ \downarrow F_{\mathcal{B}}(\phi) & \searrow (b), (c) & \uparrow F_{\mathcal{A}}(\psi) \\ F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(h) & \xrightarrow{\quad \rho(h) \quad} F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{A}}(h) \xrightarrow{\quad G_{D'} \quad} F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(h) \end{array}$$

Hence the following diagram commutes

$$\begin{array}{ccccc}
F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & & & & \\
\downarrow F_{\mathcal{B}}(\epsilon) & \searrow F_{\mathcal{B}}(\gamma) & & & \\
F_{\mathcal{A}}(i)F_{\mathcal{B}}(s)F_{\mathcal{B}}(g) & \xrightarrow{F_{\mathcal{B}}(\phi)} & F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(h)F_{\mathcal{B}}(g) & \xleftarrow{F_{\mathcal{B}}(\omega)} & F_{\mathcal{A}}(r)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) \\
\downarrow \rho(g) & & \downarrow \rho(g) & & \downarrow \rho(f) \\
F_{\mathcal{A}}(i)F_{\mathcal{B}}(s)F_{\mathcal{A}}(g) & \xrightarrow{F_{\mathcal{B}}(\phi)} & F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(h)F_{\mathcal{A}}(g) & \xleftarrow{\rho(hg)} & \\
\downarrow G_{D''} & & \downarrow \rho(h) & & \downarrow \rho(f) \\
F_{\mathcal{B}}(p)F_{\mathcal{A}}(l)F_{\mathcal{A}}(g) & \xleftarrow{F_{\mathcal{A}}(\psi)} & F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{A}}(h)F_{\mathcal{A}}(g) & \xleftarrow{F_{\mathcal{A}}(\omega)} & F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{A}}(f) \\
\downarrow F_{\mathcal{A}}(\zeta) & & \downarrow G_{D'} & & \downarrow G_{D'} \\
F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xleftarrow{F_{\mathcal{A}}(\delta)} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(h)F_{\mathcal{A}}(g) & \xleftarrow{F_{\mathcal{A}}(\omega)} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f)
\end{array}$$

Next we show that $(F_{\mathcal{A}}, F_{\mathcal{B}}, \bar{G})$ is an object of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. Axioms (a) and (a') for \bar{G} follow from axioms (c) and (c'). Let D, D' and D'' be squares as in axiom (b') for \bar{G} . Decompose it as

$$\begin{array}{ccccc}
X_1 & & & & \\
\downarrow f & \nearrow j & & & \\
W & \xrightarrow{l} & X_2 & \xrightarrow{j'} & X_3 \\
\downarrow g & \nearrow \eta & \downarrow h & \nearrow \gamma & \\
Z_1 & \xrightarrow{k} & Z_2 & \xrightarrow{k'} & X_3 \\
\downarrow q_1 & \nearrow p_2 & \downarrow q_2 & \nearrow E' & \downarrow p_3 \\
Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3
\end{array}$$

where k, k', l are 1-cells of \mathcal{A} , q_1, q_2 are 1-cells of \mathcal{B} , f, g, h are 1-cells of $\mathcal{A} \cap \mathcal{B}$, the squares E, E' and the square H containing η are 2-Cartesian in \mathcal{C} . Let E'' be the square obtained by combining E and E' , I be the square obtained by combining H and E . Since I is the outer square of the diagram

$$\begin{array}{ccccc}
W & \xrightarrow{l} & X_2 & \xlongequal{\quad} & X_2 \\
q_1 g \downarrow & J & \downarrow p_2 & \nearrow \beta & \downarrow q_2 h \\
Y_1 & \xrightarrow{i} & Y_1 & \xlongequal{\quad} & Y_2
\end{array}$$

axioms (b') and (c') imply the commutativity of the following triangle

$$\begin{array}{ccc}
F_{\mathcal{A}}(i)F_{\mathcal{B}}(q_1 g) & & \\
\downarrow G_J & \searrow G_I & \\
F_{\mathcal{B}}(p_2)F_{\mathcal{A}}(l) & \xrightarrow{F_{\mathcal{B}}(\beta)} & F_{\mathcal{B}}(q_2 h)F_{\mathcal{A}}(l)
\end{array}$$

The diagram illustrates a complex commutative structure involving functors and natural transformations. The objects are arranged in a grid-like fashion, with arrows indicating the relationships between them. The top row starts with $F_{\mathcal{A}}(i')F_{\mathcal{A}}(i)F_{\mathcal{B}}(p_1)$. A vertical arrow $F_{\mathcal{B}}(\delta)$ leads to $F_{\mathcal{A}}(i')F_{\mathcal{A}}(i)F_{\mathcal{B}}(q_1)F_{\mathcal{B}}(g)F_{\mathcal{B}}(f)$. From here, a horizontal arrow $G_{E''}$ leads to $F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(k')F_{\mathcal{A}}(k)F_{\mathcal{B}}(g)F_{\mathcal{B}}(f)$. A curved arrow $G_{D''}$ connects the top-left object to the bottom-right object $F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(j')F_{\mathcal{A}}(j)$. The diagram continues with various other objects and arrows, including $\rho(f)$, G_E , $G_{E'}$, $G_{E''}$, G_H , G_I , $F_{\mathcal{B}}(\beta')$, $F_{\mathcal{A}}(\epsilon)$, $F_{\mathcal{A}}(\gamma)$, $F_{\mathcal{A}}(\eta)$, and $F_{\mathcal{A}}(\delta)$. The bottom row ends with $F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(j')F_{\mathcal{A}}(j)$. The diagram is a representation of a complex mathematical proof or a set of relationships in a specific field of study.

One establishes axiom (b) for \bar{G} in a similar way.

Let $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G', \rho')$ be a 1-cell of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. Then $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, \bar{G}) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, \bar{G}')$ is a 1-cell of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ by 7.8. Let

$$(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}): (\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}) \Rightarrow (\epsilon'_{\mathcal{A}}, \epsilon'_{\mathcal{B}})$$

be a 2-cell of $\text{GD}'_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. Then $(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}})$ is a 2-cell of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$.

It is clear that the 2-functor defined in this way is the inverse of (7.7.1). \square

Definition 7.10. Let \mathcal{C} be a $(2, 1)$ -category, $\mathcal{A}_1, \dots, \mathcal{A}_n$ be 2-faithful subcategories of \mathcal{C} with $\text{Ob}(\mathcal{A}_1) = \dots = \text{Ob}(\mathcal{A}_n) = \text{Ob}(\mathcal{C})$, \mathcal{D} be a 2-category. Define the category $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ of *gluing data* from \mathcal{C} to \mathcal{D} , relative to $\mathcal{A}_1, \dots, \mathcal{A}_n$ as follows. An object of this category is a pair

$$((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n}),$$

where $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is an object of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$, and (F_i, F_j, G_{ij}) is an object of $\text{GD}_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$, satisfying the following condition:

(D) For $1 \leq i < j < k \leq n$ and any 2-commutative cube of the form

$$(7.10.1) \quad \begin{array}{ccccc} X' & \xrightarrow{b'} & Y' & & \\ & \searrow q' & \downarrow y & \searrow p' & \\ & Z' & \xrightarrow{a'} & W' & \\ & \downarrow z & \downarrow & \downarrow w & \\ X & \xrightarrow{b} & Y & & \\ & \searrow q & \downarrow p & \searrow & \\ & Z & \xrightarrow{a} & W & \end{array}$$

where a, b, a', b' are 1-cells of \mathcal{A}_i , p, q, p', q' are 1-cells of \mathcal{A}_j and x, y, z, w are 1-cells of \mathcal{A}_k , the following hexagon commutes

$$\begin{array}{c} F_i(a)F_j(q)F_k(x) \xrightarrow{G_{jkI'}} F_i(a)F_k(z)F_j(q') \xrightarrow{G_{ikJ}} F_k(w)F_i(a')F_j(q') \\ \Downarrow G_{ijK} \qquad \qquad \qquad \Downarrow G_{ijK'} \\ F_j(p)F_i(b)F_k(x) \xrightarrow{G_{ijJ'}} F_j(p)F_k(y)F_i(b') \xrightarrow{G_{jkI}} F_k(w)F_j(p')F_i(b') \end{array}$$

where I, I', J, J', K, K' are respectively the right, left, front, back, bottom, top faces of the cube.

A 1-cell $((F_i), (G_{ij})) \rightarrow ((F'_i), (G'_{ij}))$ of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of 1-cells $\alpha_i: F_i \rightarrow F'_i$ of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$, such that for $1 \leq i < j \leq n$,

$$(\alpha_i, \alpha_j): (F_i, F_j, G_{ij}) \rightarrow (F'_i, F'_j, G'_{ij})$$

is a 1-cell of $\text{GD}_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$.

A 2-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection

$$(\Xi_i)_{1 \leq i \leq n}: (\alpha_i)_{1 \leq i \leq n} \Rightarrow (\alpha'_i)_{1 \leq i \leq n}$$

of 2-cells $\Xi_i: \alpha_i \Rightarrow \alpha'_i$ of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$ such that $|\Xi_i| = \dots = |\Xi_n|$.

We view $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ as a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -category via the functor given by

$$\begin{aligned} ((F_i), G) &\mapsto |F_1| = \dots = |F_n|, \\ (\alpha_i) &\mapsto |\alpha_1| = \dots = |\alpha_n|, \quad (\Xi_i) \mapsto |\Xi_1| = \dots = |\Xi_n|. \end{aligned}$$

If $n = 1$, $\text{GD}_{\mathcal{A}}(\mathcal{C}, \mathcal{D}) = \text{PsFun}(\mathcal{A}, \mathcal{D})$. If $n = 2$, $\text{GD}_{\mathcal{A}_1, \mathcal{A}_2}(\mathcal{C}, \mathcal{D})$ coincides with the one defined in 7.3.

In general, this category can be described more directly as follows. An object of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a pair $((F_i)_{1 \leq i \leq n}, (G_{ijD})_{1 \leq i, j \leq n})$ (here we do *not* assume $i < j$), where $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is an object of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$ such that $|F_1| = \dots = |F_n|$,

$$G_{ijD}: F_i(a)F_j(q) \Rightarrow F_j(p)F_i(b)$$

is an invertible 2-cell of \mathcal{D} , D running over squares in \mathcal{C} of the form

$$(7.10.2) \quad \begin{array}{ccc} X & \xrightarrow{b} & Y \\ \downarrow q & \nearrow \alpha & \downarrow p \\ Z & \xrightarrow{a} & W \end{array}$$

where a, b are 1-cells of \mathcal{A}_i and p, q are 1-cells of \mathcal{A}_j , satisfying condition (D) above for all $1 \leq i, j, k \leq n$ and the following conditions:

(A) For $1 \leq i, j \leq n$ and any square D of the form

$$\begin{array}{ccc} X & \xrightarrow{b} & Y \\ \parallel & \nearrow \alpha & \parallel \\ X & \xrightarrow{a} & Y \end{array}$$

where a and b are 1-cells of \mathcal{A}_i , the following square commutes

$$\begin{array}{ccc} F_i(a) & \xRightarrow{F_i(\alpha)} & F_i(b) \\ \Downarrow & & \Downarrow \\ F_i(a)F_j(1_X) & \xRightarrow{G_{ijD}} & F_j(1_Y)F_i(b) \end{array}$$

(O) For any square D (7.10.2) with $i = j$, the following square commutes

$$\begin{array}{ccc} F_i(a)F_i(q) & \xRightarrow{G_{iiD}} & F_i(p)F_i(b) \\ \Downarrow & & \Downarrow \\ F_i(aq) & \xRightarrow{F_i(\alpha)} & F_i(pb) \end{array}$$

In fact, given $(G_{ij})_{1 \leq i < j \leq n}$, it suffices to take (O) as a definition of G_{ii} , and to put $G_{ji} = G_{ij}^*$ (7.3) for $1 \leq i < j \leq n$.

A 1-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection

$$(\alpha_i)_{1 \leq i \leq n}: ((F_i), (G_{ij})) \rightarrow ((F'_i), (G'_{ij}))$$

of 1-cells $\alpha_i: F_i \rightarrow F'_i$ of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$ such that for any square D (7.10.2), the following square commutes

$$\begin{array}{ccc} \alpha_0(W)F_i(a)F_j(q) & \xRightarrow{G_{ijD}} & \alpha_0(W)F_j(p)F_i(b) \\ \Downarrow \alpha_i(a)\alpha_j(q) & & \Downarrow \alpha_j(p)\alpha_i(b) \\ F'_i(a)F'_j(q)\alpha_0(X) & \xRightarrow{G'_{ijD}} & F'_j(p)F'_i(b)\alpha_0(X) \end{array}$$

where $\alpha_0 = |\alpha_1| = \dots = |\alpha_n|$.

It follows from this description that if σ is a permutation of $\{1, \dots, n\}$,

$$((F_i), (G_{ij})) \mapsto ((F_{\sigma(i)}), (G_{\sigma(i)\sigma(j)})), \quad (\alpha_i) \mapsto (\alpha_{\sigma(i)}), \quad (\Xi_i) \mapsto (\Xi_{\sigma(i)})$$

defines an isomorphism of $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -categories

$$\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \text{GD}_{\mathcal{A}_{\sigma(1)}, \dots, \mathcal{A}_{\sigma(n)}}(\mathcal{C}, \mathcal{D}).$$

7.11. Let $((F_i), G)$ be an object of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. For $1 \leq i, j \leq n$, (F_i, F_j, G_{ij}) is an object of $\text{GD}_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$. Let

$$\rho_{ij}: F_j|_{\mathcal{A}_i \cap \mathcal{A}_j} \rightarrow F_i|_{\mathcal{A}_i \cap \mathcal{A}_j}$$

be the 1-cell of $\text{PsFun}(\mathcal{A}_i \cap \mathcal{A}_j, \mathcal{D})$ associated to it by (7.7.1). Then $\rho_{ii} = \mathbf{1}_{F_i}$ and $\rho_{ji} = \rho_{ij}^{-1}$. We claim that ρ_{ij} has the following properties:

(E) For $1 \leq i, j, k \leq n$ and any square D (7.10.2) where a, b are 1-cells of \mathcal{A}_i and p, q are 1-cells of $\mathcal{A}_j \cap \mathcal{A}_k$, the following square commutes

$$\begin{array}{ccc} F_i(a)F_k(q) & \xrightarrow{\rho_{jk}(q)} & F_i(a)F_j(q) \\ \downarrow G_{ikD} & & \downarrow G_{ijD} \\ F_k(p)F_i(b) & \xrightarrow{\rho_{jk}(p)} & F_j(p)F_i(b) \end{array}$$

(F) (cocycle condition) For $1 \leq i, j, k \leq n$, the following triangle commutes

$$\begin{array}{ccc} F_k|_{\mathcal{A}_{ijk}} & \xrightarrow{\rho_{jk}|_{\mathcal{A}_{ijk}}} & F_j|_{\mathcal{A}_{ijk}} \\ & \searrow \rho_{ik}|_{\mathcal{A}_{ijk}} & \downarrow \rho_{ij}|_{\mathcal{A}_{ijk}} \\ & & F_i|_{\mathcal{A}_{ijk}} \end{array}$$

where $\mathcal{A}_{ijk} = \mathcal{A}_i \cap \mathcal{A}_j \cap \mathcal{A}_k$.

In fact, (E) follows from (D) applied to the cube

$$\begin{array}{ccccc} X & \xrightarrow{b} & Y & & \\ & \searrow q & & \searrow p & \\ & & Z & \xrightarrow{a} & W \\ & \downarrow q & & \downarrow p & \\ & & Z & \xrightarrow{a} & W \\ & & \parallel & & \parallel \\ & & Z & \xrightarrow{a} & W \end{array}$$

whose top and back faces are D and whose other faces have identity 2-cells. Condition (F) follows from (E) applied to the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow f & \parallel & \parallel \\ Y & \xrightarrow{f} & Y \end{array}$$

for every 1-cell f of \mathcal{A}_{ijk} .

Definition 7.12. We define a 2-category $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ as follows. An object of this 2-category is a triple $((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n}, (\rho_{ij})_{1 \leq i < j \leq n})$, where $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is an object of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$, and $(F_i, F_j, G_{ij}, \rho_{ij})$ is an object of $\text{GD}'_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$, satisfying conditions (D) for cubes with 2-Cartesian faces, (E) for pairwise distinct numbers $1 \leq i, j, k \leq n$ and 2-Cartesian squares, if we put $G_{ji} = G_{ij}^*$ and $\rho_{ji} = \rho_{ij}^{-1}$ for $1 \leq i < j \leq n$, and (F) for $1 \leq i < j < k \leq n$.

A 1-cell $((F_i), G, \rho) \rightarrow ((F'_i), G', \rho')$ of $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of 1-cells $\alpha_i: F_i \rightarrow F'_i$ of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$, such that for $1 \leq i < j \leq n$,

$$(\alpha_i, \alpha_j): (F_i, F_j, G_{ij}, \rho_{ij}) \rightarrow (F'_i, F'_j, G'_{ij}, \rho'_{ij})$$

is a 1-cell of $\text{GD}'_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$.

A 2-cell of $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection

$$(\Xi_i)_{1 \leq i \leq n}: (\alpha_i)_{1 \leq i \leq n} \Rightarrow (\alpha'_i)_{1 \leq i \leq n}$$

of 2-cells $\Xi_i: \alpha_i \Rightarrow \alpha'_i$ of $\text{PsFun}(\mathcal{A}_i, \mathcal{D})$ such that $|\Xi_1| = \dots = |\Xi_n|$.

We view $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ as a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -category via the 2-functor given by

$$\begin{aligned} ((F_i), G, \rho) &\mapsto |F_1| = \dots = |F_n|, \\ (\alpha_i) &\mapsto |\alpha_1| = \dots = |\alpha_n|, \quad (\Xi_i) \mapsto |\Xi_1| = \dots = |\Xi_n|. \end{aligned}$$

This $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -category coincides with the one defined in 7.6 if $n = 2$.

The construction 7.11 defines a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -functor

$$(7.12.1) \quad \text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}),$$

which is clearly 2-faithful. If, for all $1 \leq i < j \leq n$, \mathcal{A}_i and \mathcal{A}_j are squaring in \mathcal{C} (7.6), then (7.12.1) is 2-fully faithful.

7.13. We say that $\mathcal{A}_1, \dots, \mathcal{A}_n$ are *cubing* in \mathcal{C} if, for $1 \leq i, j \leq n$, $i \neq j$, \mathcal{A}_i and \mathcal{A}_j are squaring in \mathcal{C} , \mathcal{A}_i and $\mathcal{A}_i \cap \mathcal{A}_j$ are squaring in \mathcal{C} , and for pairwise distinct numbers $1 \leq i, j, k \leq n$, every cube (7.10.1) can be decomposed as

$$(7.13.1) \quad \begin{array}{ccccc} X' & & & & \\ \downarrow x & \nearrow f' & \delta' \Uparrow & & \\ & V' & \xrightarrow{c'} & Y' & \\ & \downarrow v & \searrow r' & \downarrow y & \searrow p' \\ X & \xrightarrow{b} & Z' & \xrightarrow{a'} & W' \\ & \downarrow \delta \Uparrow & \downarrow z & \downarrow w & \\ & V & \xrightarrow{c} & Y & \\ & \downarrow q & \searrow r & \downarrow p & \\ & & Z & \xrightarrow{a} & W \end{array}$$

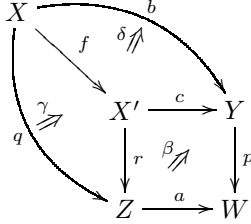
where c, c' are 1-cells of \mathcal{A}_i , r, r' are 1-cells of \mathcal{A}_j , f, f' are 1-cells of $\mathcal{A}_i \cap \mathcal{A}_j$, v is a 1-cell of \mathcal{A}_k , the bottom face L and the top face L' of the inner cube are 2-Cartesian. Note that if the right face I (resp. the front face J) of the inner cube is 2-Cartesian, so is the left face I'' (resp. the back face J'') by [IZ09, 2.12]. If $n = 1$, \mathcal{A} is always cubing in \mathcal{C} . If $n = 2$, \mathcal{A}_1 and \mathcal{A}_2 are cubing if and only if \mathcal{A}_1 and \mathcal{A}_2 are squaring in \mathcal{C} , \mathcal{A}_1 and $\mathcal{A}_1 \cap \mathcal{A}_2$ are squaring in \mathcal{C} , and \mathcal{A}_2 and $\mathcal{A}_1 \cap \mathcal{A}_2$ are squaring in \mathcal{C} . In general, one sufficient condition for $\mathcal{A}_1, \dots, \mathcal{A}_n$ to be cubing is that \mathcal{C} admits 2-fiber products, and \mathcal{A}_i is stable under 2-base change in \mathcal{C} and taking the diagonal in \mathcal{C} for $1 \leq i \leq n$.

The following generalizes 7.9.

Proposition 7.14. *Let \mathcal{C} be a $(2, 1)$ -category, $\mathcal{A}_1, \dots, \mathcal{A}_n$ be 2-faithful subcategories of \mathcal{C} with $\text{Ob}(\mathcal{A}_1) = \dots = \text{Ob}(\mathcal{A}_n) = \text{Ob}(\mathcal{C})$, \mathcal{D} be a 2-category. Assume that every 1-cell of \mathcal{C} that is an equivalence is contained in $\mathcal{A}_1 \cap \dots \cap \mathcal{A}_n$ and that $\mathcal{A}_1, \dots, \mathcal{A}_n$ are cubing. Then (7.12.1) is an isomorphism of 2-categories.*

Proof. We construct the inverse as follows. Let $((F_i), G, \rho)$ be an object of $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. For $1 \leq i, j \leq n$, $i \neq j$, let (F_i, F_j, \bar{G}_{ij}) be the image of $(F_i, F_j, G_{ij}, \rho_{ij})$ under the inverse of (7.7.1). First note that for pairwise distinct numbers $1 \leq i, j, k \leq n$, G satisfies (D) for cubes with 2-Cartesian faces and ρ satisfies (F).

Next we show that \bar{G} satisfies (E) for pairwise distinct numbers $1 \leq i, j, k \leq n$. Let D be a square (7.10.2) where a, b are 1-cells of \mathcal{A}_i and p, q are 1-cells of $\mathcal{A}_j \cap \mathcal{A}_k$, as in (E). Decompose it as



where c is a 1-cell of \mathcal{A}_i , r is a 1-cell of $\mathcal{A}_j \cap \mathcal{A}_k$, f is a 1-cell of $\mathcal{A}_i \cap \mathcal{A}_j \cap \mathcal{A}_k$, and the inner square D' is 2-Cartesian. Then the following diagram commutes

$$\begin{array}{ccccccc}
 & & \bar{G}_{ikD} & & & & \\
 F_i(a)F_k(q) & \xRightarrow{F_k(\gamma)} & F_i(a)F_k(r)F_k(f) & \xRightarrow{\rho_{ik}(f)} & F_i(a)F_k(r)F_i(f) & \xRightarrow{G_{ikD'}} & F_k(p)F_i(c)F_i(f) \xRightarrow{F_i(\delta)} F_k(p)F_i(b) \\
 \Downarrow \rho_{jk}(q) & \rho_{jk}(r)\rho_{jk}(f) \Downarrow & (F) & \Downarrow \rho_{jk}(r) & (E) & \Downarrow \rho_{jk}(p) & \Downarrow \rho_{jk}(p) \\
 F_i(a)F_j(q) & \xRightarrow{F_j(\gamma)} & F_i(a)F_j(r)F_j(f) & \xRightarrow{\rho_{ik}(f)} & F_i(a)F_j(r)F_i(f) & \xRightarrow{G_{ijD'}} & F_j(p)F_i(c)F_i(f) \xRightarrow{F_i(\delta)} F_j(p)F_i(b) \\
 & & \bar{G}_{ijD} & & & &
 \end{array}$$

For pairwise distinct numbers $1 \leq i, j, k \leq n$, we show (D) for \bar{G} by descending induction on the number m of pairs of 2-Cartesian opposite faces in the cube (7.10.1). If $m = 3$, all the faces of the cube are 2-Cartesian, so (D) for \bar{G} is identical to (D) for G . If $m < 3$, by symmetry, we may assume that either the bottom face K or the top face K' is not 2-Cartesian. Decompose the cube as (7.13.1). The inner cube has more than m pairs of 2-Cartesian opposite faces, hence (D) for \bar{G} holds for the inner cube. Therefore, the following diagram commutes

$$\begin{array}{ccccccc}
 F_i(a)F_j(q)F_k(x) & \xRightarrow{\bar{G}_{jkI'}} & F_i(a)F_k(z)F_j(q') & \xRightarrow{G_{ikJ}} & F_k(w)F_i(a')F_j(q') & & \\
 \Downarrow F_j(\gamma) & & \Downarrow F_j(\gamma') & & \Downarrow F_j(\gamma') & & \\
 F_i(a)F_j(r)F_j(f)F_k(x) & \xRightarrow{\bar{G}_{jkM}} & F_i(a)F_j(r)F_k(v)F_j(f') & \xRightarrow{\bar{G}_{jkI''}} & F_i(a)F_k(z)F_j(r')F_j(f') & \xRightarrow{\bar{G}_{ikJ}} & F_k(w)F_i(a')F_j(r')F_j(f') \\
 \Downarrow \rho_{ij}(f) & (E) & \Downarrow \rho_{ij}(f') & & \Downarrow \rho_{ij}(f') & & \\
 \bar{G}_{ijK} F_i(a)F_j(r)F_i(f)F_k(x) & \xRightarrow{\bar{G}_{ikM}} & F_i(a)F_j(r)F_k(v)F_i(f') & \xRightarrow{\bar{G}_{jkI''}} & F_i(a)F_k(z)F_j(r')F_i(f') & \xRightarrow{\bar{G}_{ikJ}} & F_k(w)F_i(a')F_j(r')F_i(f') \\
 \Downarrow G_{ijL} & & \Downarrow G_{ijL} & & (D) & & \\
 F_j(p)F_i(c)F_i(f')F_k(x) & \xRightarrow{\bar{G}_{ikM}} & F_j(p)F_i(c)F_k(v)F_i(f') & \xRightarrow{\bar{G}_{jkI''}} & F_j(p)F_k(y)F_i(c')F_i(f') & \xRightarrow{\bar{G}_{jkI}} & F_k(w)F_j(p')F_i(c')F_i(f') \\
 \Downarrow F_i(\delta) & & \Downarrow F_i(\delta') & & \Downarrow F_i(\delta') & & \\
 F_j(p)F_i(b)F_k(x) & \xRightarrow{\bar{G}_{ikJ'}} & F_j(p)F_k(y)F_i(b') & \xRightarrow{\bar{G}_{jkI}} & F_k(w)F_j(p')F_i(b') & &
 \end{array}$$

where M is the square $X'V'XV$.

Any 1-cell

$$(\alpha_i): ((F_i), G, \rho) \rightarrow ((F'_i), G', \rho')$$

of $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a 1-cell $((F_i), \bar{G}) \rightarrow ((F'_i), \bar{G}')$ of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. Any 2-cell $(\Xi_i): (\alpha_i) \Rightarrow (\alpha'_i)$ of $\text{GD}'_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a 2-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$.

It is clear that the 2-functor defined in this way is the inverse of (7.12.1). \square

If $n \geq 2$ and \mathcal{B} is a 2-faithful subcategory of \mathcal{C} containing \mathcal{A}_1 and \mathcal{A}_2 , the $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -functor (7.3.3)

$$(7.14.1) \quad \text{PsFun}(\mathcal{B}, \mathcal{D}) \rightarrow \text{GD}_{\mathcal{A}_1, \mathcal{A}_2}(\mathcal{B}, \mathcal{D})$$

induces a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -functor

$$(7.14.2) \quad \text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

as follows. For any object $(F_{\mathcal{B}}, (F_i)_{3 \leq i \leq n}, (G_{\mathcal{B}j})_{3 \leq j \leq n}, (G_{ij})_{3 \leq i < j \leq n})$ of $\text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, let (F_1, F_2, G_{12}) be the image of $F_{\mathcal{B}}$ under (7.14.1). Then $F_1 = F_{\mathcal{B}}|_{\mathcal{A}_1}$ and $F_2 = F_{\mathcal{B}}|_{\mathcal{A}_2}$. For $i = 1, 2$ and $3 \leq j \leq n$, let G_{ij} be the restriction of $G_{\mathcal{B}j}$. Then $((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n})$ is an object of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. For any 1-cell $(\alpha_{\mathcal{B}}, (\alpha_i)_{3 \leq i \leq n})$ of $\text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, let $(\alpha_1, \alpha_2) = (\alpha_{\mathcal{B}}|_{\mathcal{A}_1}, \alpha_{\mathcal{B}}|_{\mathcal{A}_2})$ be the image of $\alpha_{\mathcal{B}}$ under (7.14.1). Then $(\alpha_i)_{1 \leq i \leq n}$ is a 1-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. For any 2-cell $(\Xi_{\mathcal{B}}, (\Xi_i)_{3 \leq i \leq n})$ of $\text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, let $(\Xi_1, \Xi_2) = (\Xi_{\mathcal{B}}|_{\mathcal{A}_1}, \Xi_{\mathcal{B}}|_{\mathcal{A}_2})$ be the image of $\Xi_{\mathcal{B}}$ under (7.14.1). Then $(\Xi_i)_{1 \leq i \leq n}$ is a 2-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. The 2-functor (7.14.2) is 2-faithful if \mathcal{A} and \mathcal{B} generate \mathcal{C} (7.4).

The case $\mathcal{B} = \mathcal{A}_2$ is simple:

Proposition 7.15. *Let \mathcal{C} be a $(2, 1)$ -category, $n \geq 2$, $\mathcal{A}_1, \dots, \mathcal{A}_n$ be 2-faithful subcategories of \mathcal{C} with $\text{Ob}(\mathcal{A}_1) = \dots = \text{Ob}(\mathcal{A}_n) = \text{Ob}(\mathcal{C})$, $\mathcal{A}_1 \subset \mathcal{A}_2$, \mathcal{D} be a 2-category. Then the 2-functor (7.14.2)*

$$Q: \text{GD}_{\mathcal{A}_2, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

is a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -quasi-inverse to the projection 2-functor P .

Proof. It is clear that PQ is the identity functor. For any object $((F_i), (G_{ij}))$ of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, $\rho_{12}: F_2|_{\mathcal{A}_1} \rightarrow F_1$ defines an invertible 1-cell

$$(\rho_{12}, \mathbf{1}_{F_2}, \dots, \mathbf{1}_{F_n}): QP((F_i), (G_{ij})) \rightarrow ((F_i), (G_{ij})).$$

This gives a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -natural equivalence $QP \Rightarrow \mathbf{1}$. \square

The following generalizes 7.5.

Theorem 7.16. *Let \mathcal{C} be a $(2, 1)$ -category, $n \geq 2$, $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ be 2-faithful subcategories (7.1) of \mathcal{C} with $\text{Ob}(\mathcal{A}_1) = \dots = \text{Ob}(\mathcal{A}_n) = \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{C})$, $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}$, \mathcal{D} be a 2-category. Assume the following*

- (i) *Every 1-cell of \mathcal{B} is isomorphic to pa for some 1-cell a of \mathcal{A}_1 and some 1-cell p of \mathcal{A}_2 .*
- (ii) *2-fiber products exist in \mathcal{A}_2 and are 2-fiber products in \mathcal{B} .*
- (iii) *For $3 \leq i \leq n$, 2-base changes of 1-cells of \mathcal{A}_i by 1-cells of \mathcal{A}_2 in \mathcal{C} exist and are 1-cells of \mathcal{A}_i .*
- (iv) *For $i = 1, 2$ and $3 \leq j \leq n$, 1-cells of \mathcal{A}_i are stable under 2-base change by 1-cells of \mathcal{A}_j in \mathcal{C} , whenever the 2-base change exists.*
- (v) *For any $3 \leq i \leq n$, \mathcal{B} and \mathcal{A}_i are squaring in \mathcal{C} (7.8).*
- (vi) *For any $3 \leq i \leq n$, $\mathcal{A}_1 \cap \mathcal{A}_i$ and $\mathcal{A}_2 \cap \mathcal{A}_i$ generate $\mathcal{B} \cap \mathcal{A}_i$.*

Then (7.14.2) is 2-fully faithful. Moreover, (7.14.2) is a $2\text{-Fun}(\text{Ob}(\mathcal{C}), \mathcal{D})$ -equivalence (7.1) under the following additional assumptions

- (iv') For $i = 1, 2$ and $3 \leq j \leq n$, \mathcal{A}_i and \mathcal{A}_j are squaring.
- (v') $\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n$ are cubing in \mathcal{C} (7.13).
- (vi') For any $3 \leq i \leq n$, every 1-cell of $\mathcal{B} \cap \mathcal{A}_i$ is isomorphic to pa for some 1-cell a of $\mathcal{A}_1 \cap \mathcal{A}_i$ and some 1-cell p of $\mathcal{A}_2 \cap \mathcal{A}_i$.
- (vii) For any $3 \leq i \leq n$, 2-fiber products exist in $\mathcal{A}_2 \cap \mathcal{A}_i$ and are 2-fiber products in $\mathcal{B} \cap \mathcal{A}_i$.
- (viii) For $3 \leq i \leq n$, 2-base changes of 1-cells of $\mathcal{A}_2 \cap \mathcal{A}_i$ by 1-cells of \mathcal{A}_1 in \mathcal{C} exist.
- (ix) For $i = 1, 2$ and $3 \leq j \leq n$, 1-cells of $\mathcal{A}_i \cap \mathcal{A}_j$ are stable under 2-base change by 1-cells of \mathcal{B} and by 1-cells of \mathcal{A}_j in \mathcal{C} , whenever the 2-base change exists.
- (x) For $3 \leq i < j \leq n$, $\mathcal{A}_1 \cap \mathcal{A}_i \cap \mathcal{A}_j$ and $\mathcal{A}_2 \cap \mathcal{A}_i \cap \mathcal{A}_j$ generate $\mathcal{B} \cap \mathcal{A}_i \cap \mathcal{A}_j$.

Note that (v') implies (v), (vi') implies (vi). If \mathcal{C} admits 2-fiber products, and \mathcal{A}_i , $1 \leq i \leq n$ and \mathcal{B} are stable under 2-base change in \mathcal{C} and taking the diagonal in \mathcal{C} , then all the conditions, except (i), (vi), (vi') and (x), are automatic.

Proof. The case $n = 2$ is 7.5. In the sequel we assume $n \geq 3$. Then (iii) and (iv) imply that $\mathcal{A}_1 \cap \dots \cap \mathcal{A}_n$ contains all 1-cells of \mathcal{C} that are equivalences.

Since \mathcal{A}_1 and \mathcal{A}_2 generate \mathcal{B} by (i), (7.14.2) is 2-faithful. Let $(F_{\mathcal{B}}, (F_i), (G_{\mathcal{B}j}), (G_{ij}))$ and $(F'_{\mathcal{B}}, (F'_i), (G'_{\mathcal{B}j}), (G'_{ij}))$ be objects of $\text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ and

$$(\epsilon_i): ((F_i), G) \rightarrow ((F'_i), G')$$

be a 1-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ whose source and target are respectively the images of $(F_{\mathcal{B}}, (F_i), (G_{\mathcal{B}j}), (G_{ij}))$ and $(F'_{\mathcal{B}}, (F'_i), (G'_{\mathcal{B}j}), (G'_{ij}))$ under (7.14.2). By (i), (ii) and 7.5, (ϵ_1, ϵ_2) has a unique inverse image $\epsilon_{\mathcal{B}}: F_{\mathcal{B}} \rightarrow F'_{\mathcal{B}}$ in $\text{PsFun}(\mathcal{B}, \mathcal{D})$. We claim that

$$(\epsilon_{\mathcal{B}}, (\epsilon_i)): (F_{\mathcal{B}}, (F_i), G) \rightarrow (F'_{\mathcal{B}}, (F'_i), G')$$

is a 1-cell of $\text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. By (v), it suffices to check the compatibility with $G_{\mathcal{B}iD}$ for 2-Cartesian diagrams D , and with $\rho_{\mathcal{B}i}$, $3 \leq i \leq n$. By (vi), the square

$$\begin{array}{ccc} F_i | \mathcal{B} \cap \mathcal{A}_i & \xrightarrow{\rho_{\mathcal{B}i}} & F_{\mathcal{B}} | \mathcal{B} \cap \mathcal{A}_i \\ \downarrow \epsilon_i | \mathcal{B} \cap \mathcal{A}_i & & \downarrow \epsilon_{\mathcal{B}} | \mathcal{B} \cap \mathcal{A}_i \\ F'_i | \mathcal{B} \cap \mathcal{A}_i & \xrightarrow{\rho'_{\mathcal{B}i}} & F'_{\mathcal{B}} | \mathcal{B} \cap \mathcal{A}_i \end{array}$$

commutes because it commutes after restriction to $\mathcal{A}_1 \cap \mathcal{A}_i$ and to $\mathcal{A}_2 \cap \mathcal{A}_i$. By (i) and (iii), any 2-Cartesian square D (7.10.2) with a, b in \mathcal{B} and p, q in \mathcal{A}_i can be completed into a commutative prism with 2-Cartesian joining faces

(7.16.1)

$$\begin{array}{ccccc} & & U & & \\ & b_1 \nearrow & \uparrow \delta & \nwarrow b_2 & \\ X & \xrightarrow{b} & Y & & \\ & D_1 \searrow & \downarrow r & \swarrow D_2 & \\ & & V & & \\ & a_1 \nearrow & \uparrow \gamma & \nwarrow a_2 & \\ Z & \xrightarrow{a} & W & & \\ & q \downarrow & & \downarrow p & \end{array}$$

with a_1 in \mathcal{A}_1 and a_2 in \mathcal{A}_2 . By (iii), r is in \mathcal{A}_i . By (iv), b_1 is in \mathcal{A}_1 and b_2 is in \mathcal{A}_2 . Then the following diagram commutes

$$\begin{array}{ccc}
\epsilon_0(W)F_{\mathcal{B}}(a)F_i(q) & \xRightarrow{\epsilon_{\mathcal{B}}(a)\epsilon_i(q)} & F'_{\mathcal{B}}(a)F'_i(q)\epsilon_0(X) \\
\Downarrow F'_{\mathcal{B}}(\gamma) & & \Downarrow F_{\mathcal{B}}(\gamma) \\
\epsilon_0(W)F_2(a_2)F_1(a_1)F_i(q) & \xRightarrow{\epsilon_2(a_2)\epsilon_1(a_1)\epsilon_i(q)} & F'_2(a_2)F'_1(a_1)F'_i(q)\epsilon_0(X) \\
\Downarrow G_{1iD_1} & & \Downarrow G'_{1iD_1} \\
\epsilon_0(W)F_2(a_2)F_i(r)F_1(b_1) & \xRightarrow{\epsilon_2(a_2)\epsilon_i(r)\epsilon_1(b_1)} & F'_2(a_2)F'_i(r)F'_1(b_1)\epsilon_0(X) \\
\Downarrow G_{2iD_2} & & \Downarrow G'_{2iD_2} \\
\epsilon_0(W)F_i(p)F_2(b_2)F_1(b_1) & \xRightarrow{\epsilon_i(p)\epsilon_2(b_2)\epsilon_1(b_1)} & F'_i(p)F'_2(b_2)F'_1(b_1)\epsilon_0(X) \\
\Downarrow F_{\mathcal{B}}(\delta)^{-1} & & \Downarrow F'_{\mathcal{B}}(\delta)^{-1} \\
\epsilon_0(W)F_i(p)F_{\mathcal{B}}(b) & \xRightarrow{\epsilon_i(p)\epsilon_{\mathcal{B}}(b)} & F'_i(p)F'_{\mathcal{B}}(b)\epsilon_0(X)
\end{array}$$

$G_{\mathcal{B}iD}$ (left curved arrow) $G'_{\mathcal{B}iD}$ (right curved arrow)

Here $\epsilon_0 = |\epsilon_1| = \dots = |\epsilon_n|$. The image of $(\epsilon_{\mathcal{B}}, (\epsilon_i))$ under (7.14.2) is (ϵ_i) . Therefore (7.14.2) is 2-fully faithful.

It remains to show that, under the additional assumptions, every object $((F_i), G)$ of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is in the 2-Fun($\text{Ob}(\mathcal{C}), \mathcal{D}$)-essential image of (7.14.2). We construct an object

$$(F_{\mathcal{B}}, (F_i), (G_{\mathcal{B}j}), (G_{ij}), (\rho_{\mathcal{B}j}), (\rho_{ij}))$$

of $\text{GD}'_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ as follows. By 7.5, there exists an object $F_{\mathcal{B}}$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ with $|F_{\mathcal{B}}| = |F_1| = |F_2|$ equipped with an invertible 1-cell $(\epsilon_1, \epsilon_2): (F_1, F_2, G_{12}) \rightarrow (F_{\mathcal{B}}|_{\mathcal{A}_1}, F_{\mathcal{B}}|_{\mathcal{A}_2}, F_{\mathcal{B}})$, where the target is the image of $F_{\mathcal{B}}$ under (7.14.1), with $|\epsilon_1| = |\epsilon_2| = \mathbf{1}_{|F_{\mathcal{B}}|}$. For $3 \leq i \leq n$, by (vi'), (vii) and 7.5, the 2-functor

$$\text{PsFun}(\mathcal{B} \cap \mathcal{A}_i, \mathcal{D}) \rightarrow \text{GD}_{\mathcal{A}_1 \cap \mathcal{A}_i, \mathcal{A}_2 \cap \mathcal{A}_i}(\mathcal{B} \cap \mathcal{A}_i, \mathcal{D})$$

is a 2-Fun($\text{Ob}(\mathcal{C}), \mathcal{D}$)-equivalence. For any square D (7.10.2) with a, b in $\mathcal{A}_1 \cap \mathcal{A}_i$ and p, q in $\mathcal{A}_2 \cap \mathcal{A}_i$, the following diagram commutes

$$\begin{array}{ccccc}
F_i(a)F_i(q) & \xRightarrow{\rho_{i1}(a)} & F_1(a)F_i(q) & \xRightarrow{\rho_{i2}(q)} & F_1(a)F_2(q) \\
\Downarrow F_i(\alpha) & (c') & \Downarrow G_{1iD} & (E) & \Downarrow G_{12D} \\
F_i(p)F_i(b) & \xRightarrow{\rho_{i1}(b)} & F_i(p)F_1(b) & \xRightarrow{\rho_{i2}(q)} & F_2(p)F_1(b)
\end{array}$$

In other words,

$$(\rho_{i1}, \rho_{i2}): (F_i|_{\mathcal{A}_1 \cap \mathcal{A}_i}, F_i|_{\mathcal{A}_2 \cap \mathcal{A}_i}, F_i|_{\mathcal{B} \cap \mathcal{A}_i}) \rightarrow (F_1|_{\mathcal{A}_1 \cap \mathcal{A}_i}, F_2|_{\mathcal{A}_2 \cap \mathcal{A}_i}, G_{12})$$

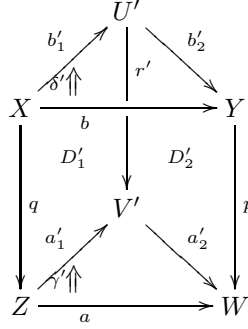
is a 1-cell of $\text{GD}_{\mathcal{A}_1 \cap \mathcal{A}_i, \mathcal{A}_2 \cap \mathcal{A}_i}(\mathcal{B} \cap \mathcal{A}_i, \mathcal{D})$. Composing it with

$$(\epsilon_1|_{\mathcal{A}_1 \cap \mathcal{A}_i}, \epsilon_2|_{\mathcal{A}_2 \cap \mathcal{A}_i}),$$

we obtain a 1-cell $\rho_{\mathcal{B}i}: F_i|_{\mathcal{B} \cap \mathcal{A}_i} \rightarrow F_{\mathcal{B}}|_{\mathcal{B} \cap \mathcal{A}_i}$ of $\text{PsFun}(\mathcal{B} \cap \mathcal{A}_i, \mathcal{D})$. For any 2-Cartesian square D (7.10.2) with a, b in \mathcal{B} and p, q in \mathcal{A}_i , complete it as (7.16.1) and define $G_{\mathcal{B}iD}$ to be the unique invertible 2-cell making the following diagram commute

$$\begin{array}{ccc}
F_{\mathcal{B}}(a)F_i(q) & \xRightarrow{G_{\mathcal{B}iD}} & F_i(p)F_{\mathcal{B}}(b) \\
\Downarrow F_{\mathcal{B}}(\gamma) & & \Downarrow F_{\mathcal{B}}(\delta) \\
F_2(a_2)F_1(a_1)F_i(q) & \xRightarrow{G_{1iD_1}} F_2(a_2)F_i(r)F_1(b_1) \xRightarrow{G_{2iD_2}} & F_i(p)F_2(b_2)F_1(b_1)
\end{array}$$

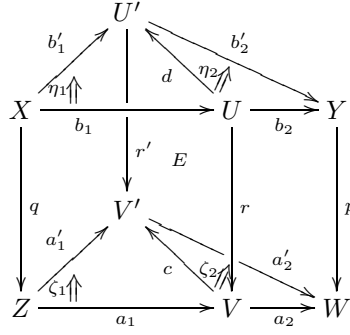
This does not depend on the choice of the prism. To see this, let



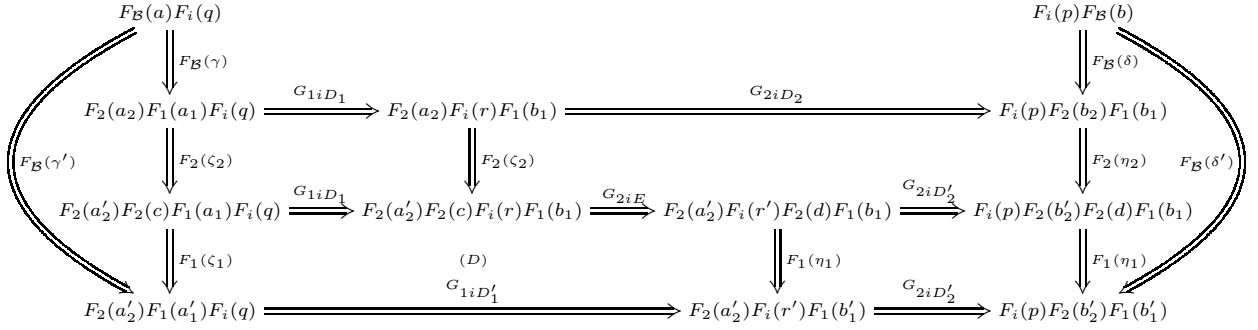
be another prism with a'_1 in \mathcal{A}_1 , a'_2 in \mathcal{A}_2 , D'_1 and D'_2 2-Cartesian. As in the proof of 7.5, we may assume that there exists a 1-cell of compactifications

$$(c, \zeta_1, \zeta_2): (V, a_1, a_2, \gamma) \rightarrow (V', a'_1, a'_2, \gamma'),$$

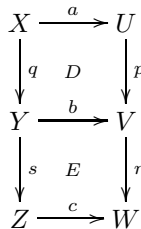
c in \mathcal{A}_2 . Then the second prism can be decomposed into the first prism and the following



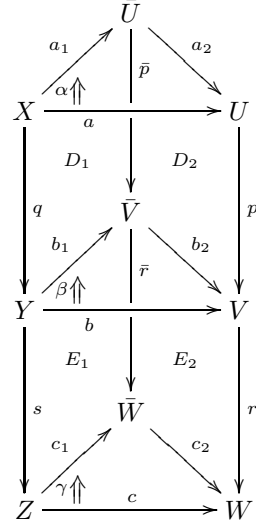
Then the following diagram commutes



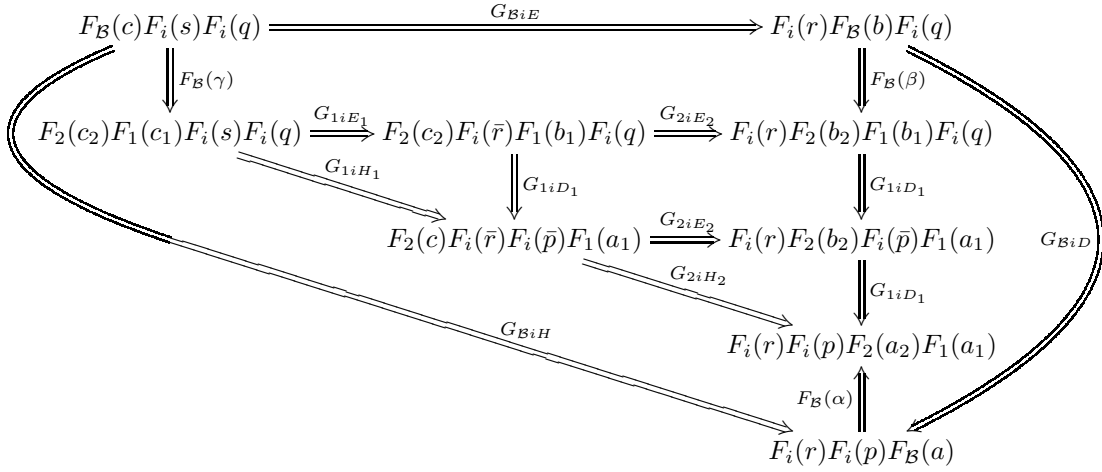
Let $3 \leq i \leq n$. We first check axioms (b), (b'), (c) and (c') for $(F_B, F_i, G_{Bi}, \rho_{Bi})$ and 2-Cartesian squares. To check (b), let H be the outer square of the diagram with 2-Cartesian squares



whose horizontal 1-cells are in \mathcal{B} and vertical 1-cells are in \mathcal{A}_i . It can be completed into a diagram with 2-Cartesian squares



with c_1 in \mathcal{A}_1 , c_2 in \mathcal{A}_2 . Then the following diagram commutes

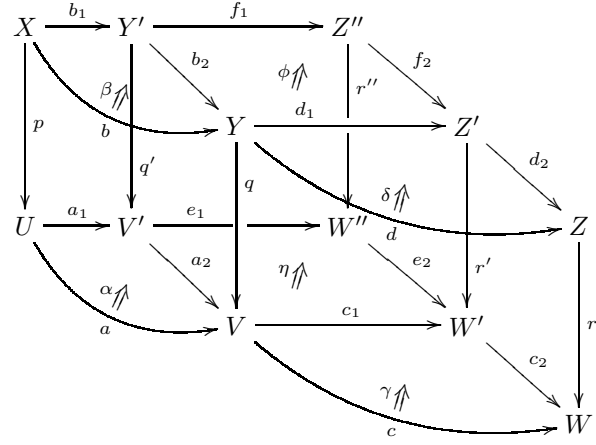


Here H_j is the square obtained by combining D_j and E_j , $j = 1, 2$. To check (b'), let I be the outer square of the diagram with 2-Cartesian squares

$$\begin{array}{ccccc} X & \xrightarrow{b} & Y & \xrightarrow{d} & Z \\ \downarrow p & D & \downarrow q & E & \downarrow r \\ U & \xrightarrow{a} & V & \xrightarrow{c} & W \end{array}$$

whose horizontal 1-cells are in \mathcal{B} and vertical 1-cells are in \mathcal{A}_i . It can be completed into a diagram with

2-Cartesian squares



where a_1, c_1, e_1 are 1-cells of \mathcal{A}_1 , a_2, c_2, e_2 are 1-cells of \mathcal{A}_2 . Let $D_1 = XY'UV'$, $E_2 = Z'ZW'W$, and let D_2, E_1, H_1, H_2 be respectively the left, front, back, right faces of the cube. Then the following diagram

[illegible]

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category, because of (b) and (b'). If the square is a composition of two squares, it suffices to check (c) and (c') for the two squares, again by (b) and (b'). By (vi'), (viii) and (ix), we are then reduced to the same axioms for G_{1i} and G_{2i} .

To check (D) for cubes with 2-Cartesian faces, let $3 \leq i < j \leq n$ and (7.10.1) be such a cube with a, b, a', b' in \mathcal{B} , p, q, p', q' in \mathcal{A}_i , x, y, z, w in \mathcal{A}_j . By (b) and (b'), we may replace any 1-cell in the cube by a 1-cell isomorphic to it in the same category. Thus we may assume that the cube can be decomposed into the following diagram with 2-Cartesian squares

$$\begin{array}{ccccccc}
X' & \xrightarrow{b'_1} & \bar{Y}' & \xrightarrow{b'_2} & Y' & & \\
& \searrow q' & \downarrow \bar{y} & \searrow \bar{p}' & \downarrow y & \searrow p' & \\
& & Z' & \xrightarrow{a'_1} & \bar{W}' & \xrightarrow{a'_2} & W' \\
& & \downarrow z & \downarrow \bar{w} & \downarrow & & \downarrow w \\
X & \xrightarrow{b_1} & \bar{Y} & \xrightarrow{b_2} & Y & & \\
& \searrow q & \downarrow \bar{p} & \downarrow & \downarrow p & & \\
& & Z & \xrightarrow{a_1} & \bar{W} & \xrightarrow{a_2} & W
\end{array}$$

where a_1 is in \mathcal{A}_1 and a_2 is in \mathcal{A}_2 . Denote the bottom, top, front, back faces of the left (resp. right) cube by K_k, K'_k, J_k, J'_k , $k = 1$ (resp. $k = 2$), and the common face of the two cubes by \bar{I} . Then the following diagram commutes

$$\begin{array}{ccccc}
& & \xrightarrow{G_{\mathcal{B}iK}} & & \\
F_2(a_2)F_1(a_1)F_i(p)F_j(x) & \xrightarrow{G_{1iK_1}} & F_2(a_2)F_i(\bar{p})F_1(b_1)F_j(x) & \xrightarrow{G_{2iK_2}} & F_i(p)F_2(b_2)F_1(b_1)F_j(x) \\
\downarrow G_{ijI'} & & \downarrow G_{1jJ'_1} & & \downarrow G_{1jJ'_1} \\
F_2(a_2)F_1(a_1)F_j(z)F_i(q') & & F_2(a_2)F_i(\bar{p})F_j(\bar{y})F_1(b'_1) & \xrightarrow{G_{2iK_2}} & F_i(p)F_2(b_2)F_j(\bar{y})F_1(b'_1) \\
\downarrow G_{1jJ_1} & & \downarrow G_{ij\bar{I}} & & \downarrow G_{2jJ'_2} \\
F_2(a_2)F_j(\bar{w})F_1(a'_1)F_i(q') & \xrightarrow{G_{1iK'_1}} & F_2(a_2)F_j(\bar{w})F_i(\bar{p}')F_1(b'_1) & & F_i(p)F_j(y)F_2(b'_2)F_1(b'_1) \\
\downarrow G_{2jJ_2} & & \downarrow G_{2jJ_2} & & \downarrow G_{ij\bar{I}} \\
F_j(w)F_2(a'_2)F_1(a'_1)F_i(q') & \xrightarrow{G_{1iK'_1}} & F_j(w)F_2(a'_2)F_i(\bar{p}')F_1(b'_1) & \xrightarrow{G_{2iK'_2}} & F_j(w)F_i(p')F_2(b'_2)F_1(b'_1) \\
& & \xrightarrow{G_{\mathcal{B}iK'}} & &
\end{array}$$

The proof of (E) for 2-Cartesian squares is similar to the proof of (c) and (c'). By (x), to check (F), it suffices to do so after restriction to $\mathcal{A}_1 \cap \mathcal{A}_i \cap \mathcal{A}_j$ and to $\mathcal{A}_2 \cap \mathcal{A}_i \cap \mathcal{A}_j$, which follows from (F) for $(\rho_{ij})_{1 \leq i < j \leq n}$.

By (v') and 7.14, the object of $\text{GD}'_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ we constructed defines an object $(F_{\mathcal{B}}, (F_i), (G_{\mathcal{B}j}), (G_{ij}))$ of $\text{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. It remains to show that

$$(\epsilon_1, \epsilon_2, \mathbf{1}_{F_3}, \dots, \mathbf{1}_{F_n}) : (F_1, \dots, F_n, G) \rightarrow (F_{\mathcal{B}}|_{\mathcal{A}_1}, F_{\mathcal{B}}|_{\mathcal{A}_2}, F_3, \dots, F_n, G)$$

is an invertible 1-cell of $\text{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. By (iv'), it suffices to check the compatibility with ρ and with G for 2-Cartesian squares, which is clear from the construction. \square

7.17. Let \mathcal{D} be a 2-category. We denote by \mathcal{D}^{op} the 2-category obtained by reversing 1-cells and 2-cells. In other words, $\text{Ob}(\mathcal{D}^{\text{op}}) = \text{Ob}(\mathcal{D})$ and, for any pair of objects X and Y of \mathcal{D} , $\mathcal{D}^{\text{op}}(Y, X) = \mathcal{D}(X, Y)^{\text{op}}$.

Define a 2-category \mathcal{D}^{adj} with $\text{Ob}(\mathcal{D}^{\text{adj}}) = \text{Ob}(\mathcal{D})$ as follows. For any pair of objects X and Y of \mathcal{D} , let $\mathcal{D}^{\text{adj}}(X, Y)$ be the category of adjoint pairs from X to Y . Hence a 1-cell $X \rightarrow Y$ in \mathcal{D}^{adj} is a

quadruple (f, g, η, ϵ) consisting of 1-cells $f: X \rightarrow Y$, $g: Y \rightarrow X$ and 2-cells $\eta: \mathbf{1}_Y \Rightarrow fg$, $\epsilon: gf \Rightarrow \mathbf{1}_X$ of \mathcal{D} such that the following triangles commute

$$\begin{array}{ccc} f & \xRightarrow{\eta} & fgf \\ \searrow \mathbf{1}_f & & \downarrow \epsilon \\ & & f \end{array} \quad \begin{array}{ccc} g & \xRightarrow{\eta} & gfg \\ \searrow \mathbf{1}_g & & \downarrow \epsilon \\ & & g \end{array}$$

The composition of $(f_1, g_1, \eta_1, \epsilon_1): X \rightarrow Y$ and $(f_2, g_2, \eta_2, \epsilon_2): Y \rightarrow Z$ is

$$(f_2 f_1, g_1 g_2, \eta_1 \eta_2, \epsilon_1 \epsilon_2): X \rightarrow Z,$$

where $\eta_1 \eta_2$ is the composition

$$\mathbf{1}_Z \xRightarrow{\eta_2} f_2 g_2 \xRightarrow{\eta_1} f_2 f_1 g_1 g_2$$

and $\epsilon_1 \epsilon_2$ is the composition

$$g_1 g_2 f_1 f_2 \xRightarrow{\epsilon_2} g_1 f_1 \xRightarrow{\epsilon_1} \mathbf{1}_X.$$

The identity 1-cell of an object X is $(\mathbf{1}_X, \mathbf{1}_X, \mathbf{1}_{\mathbf{1}_X}, \mathbf{1}_{\mathbf{1}_X})$. A 2-cell $(f, g, \eta, \epsilon) \Rightarrow (f', g', \eta', \epsilon')$ of \mathcal{D}^{adj} is a pair (α, β) of 2-cells $\alpha: f \Rightarrow f'$ and $\beta: g' \Rightarrow g$ of \mathcal{D} such that the following squares commute

$$\begin{array}{ccc} \mathbf{1}_Y & \xRightarrow{\eta} & fg \\ \downarrow \eta' & & \downarrow \alpha \\ f'g' & \xRightarrow{\beta} & f'g \end{array} \quad \begin{array}{ccc} g'f & \xRightarrow{\alpha} & g'f' \\ \downarrow \beta & & \downarrow \epsilon' \\ gf & \xRightarrow{\epsilon} & \mathbf{1}_X \end{array}$$

The projection 2-functors $P_1: \mathcal{D}^{\text{adj}} \rightarrow \mathcal{D}$ and $P_2: \mathcal{D}^{\text{adj}} \rightarrow \mathcal{D}^{\text{op}}$ are pseudo-faithful (7.1).

Let \mathcal{C} be a 2-category. We do not assume that \mathcal{C} is a $(2, 1)$ -category. Then P_1 and P_2 induce pseudo-faithful 2-functors

$$P_1: \text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{adj}}) \rightarrow \text{PsFun}(\mathcal{C}, \mathcal{D}), \quad P_2: \text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{adj}}) \rightarrow \text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{op}}).$$

An object F of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ (resp. $\text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{op}})$) is in the image of P_1 (resp. P_2) if and only if for every 1-cell a of \mathcal{C} , $F(a)$ can be completed into an adjoint pair $(F(a), g, \eta, \epsilon)$ (resp. $(f, F(a), \eta, \epsilon)$).

7.18. Fix a pseudo-functor $F: \mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$, a 2-faithful subcategory \mathcal{B} of \mathcal{C} and a pseudo-functor $B: \mathcal{B} \rightarrow \mathcal{D}^{\text{adj}}$ such that $P_2(B) = F|_{\mathcal{B}}$. We denote F by

$$f \mapsto f^*, \quad \alpha \mapsto \alpha^*,$$

and the pseudo-functor $R = P_1(B): \mathcal{B} \rightarrow \mathcal{D}$ by

$$p \mapsto p_*, \quad \alpha \mapsto \alpha_*.$$

Let D be a square in \mathcal{C}

$$(7.18.1) \quad \begin{array}{ccc} X & \xrightarrow{j} & Y \\ \downarrow q & \searrow \alpha & \downarrow p \\ Z & \xrightarrow{i} & W \end{array}$$

where p and q are 1-cells of \mathcal{B} . The base change map B_D is by definition the following 2-cell of \mathcal{D}

$$(7.18.2) \quad i^* p_* \xRightarrow{\eta_q} q_* q^* i^* p_* \xRightarrow{\quad} q_*(iq)^* p_* \xRightarrow{\alpha^*} q_*(pj)^* p_* \xRightarrow{\quad} q_* j^* p^* p_* \xRightarrow{\epsilon_p} q_* j^*.$$

If i and j are also 1-cells of \mathcal{B} , then B_D is also the composition

$$i^*p_* \xRightarrow{\eta_j} i^*p_*j_*j^* \xRightarrow{\alpha_*} i^*(pj)_*j^* \xRightarrow{\alpha_*} i^*(iq)_*j^* \xRightarrow{\alpha_*} i^*i_*q_*j^* \xRightarrow{\epsilon_i} q_*j^* .$$

In fact, the following diagram commutes

$$\begin{array}{ccccccc}
& & & i^*p_* & \xrightarrow{\eta_q} & q_*q^*i^*p_* & \\
& & \nearrow 1 & \uparrow \epsilon_i & & \downarrow \alpha^* & \\
i^*p_* & \xrightarrow{\eta_i} & i^*i_*i^*p_* & \xrightarrow{\eta_q} & i^*i_*q_*q^*i^*p_* & & \\
\downarrow \eta_p & & & & \downarrow \alpha^* & & \\
i^*p_*p^*p_* & \xrightarrow{\eta_j} & i^*p_*j_*j^*p^*p_* & \xrightarrow{\alpha_*} & i^*i_*q_*j^*p^*p_* & \xrightarrow{\epsilon_i} & q_*j^*p^*p_* \\
\downarrow \epsilon_p & & & & \downarrow \epsilon_p & & \downarrow \epsilon_p \\
i^*p_* & \xrightarrow{\eta_j} & i^*p_*j_*j^* & \xrightarrow{\alpha_*} & i^*i_*q_*j^* & \xrightarrow{\epsilon_i} & q_*j^*
\end{array}$$

Proposition 7.19. (i) Let D, D', D'' be respectively the upper, lower and outer squares of the diagram in \mathcal{C}

$$\begin{array}{ccc}
X_1 & \xrightarrow{i_1} & Y_1 \\
\downarrow q & \not\downarrow \alpha & \downarrow p \\
X_2 & \xrightarrow{i_2} & Y_2 \\
\downarrow q' & \not\downarrow \alpha' & \downarrow p' \\
X_3 & \xrightarrow{i_3} & Y_3
\end{array}$$

where the vertical arrows are 1-cells of \mathcal{B} . Then the following diagram commutes

$$\begin{array}{ccccc}
i_3^*p'_*p_* & \xRightarrow{B_{D'}} & q'_*i_2^*p_* & \xRightarrow{B_D} & q'_*q_*i_1^* \\
\Downarrow & & & & \Downarrow \\
i_3^*(p'p)_* & \xRightarrow{B_{D''}} & (q'q)_*i_1^* & &
\end{array}$$

(ii) Let D, D', D'' be respectively the left, right and outer squares of the diagram in \mathcal{C}

$$\begin{array}{ccccc}
X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\
\downarrow p_1 & \not\downarrow \alpha & \downarrow p_2 & \not\downarrow \alpha' & \downarrow p_3 \\
Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3
\end{array}$$

where the vertical arrows are 1-cells of \mathcal{B} . Then the following diagram commutes

$$\begin{array}{ccccc}
j^*j'^*p_{3*} & \xRightarrow{B_D} & j^*p_{2*}a'^* & \xRightarrow{B_{D'}} & p_{1*}i'^*i^* \\
\Uparrow & & & & \Uparrow \\
(j'j)^*p_{3*} & \xRightarrow{B_{D''}} & p_{1*}(i'i)^* & &
\end{array}$$

Proof. (i) The following diagram commutes

$$\begin{array}{ccccccc}
 & & & B_{D'} & & & \\
 i_3^* p'_* p_* & \xRightarrow{\eta_{q'}} & q'_* q'^* i_3^* p'_* p_* & \xRightarrow{\alpha'^*} & q'_* i_2^* p'^* p'_* p_* & \xRightarrow{\epsilon_{p'}} & q'_* i_2^* p_* \\
 & \searrow \eta_q & \downarrow \eta_q & & \downarrow \eta_q & & \downarrow \eta_q \\
 & & q'_* q_* q'^* i_3^* p'_* p_* & \xRightarrow{\alpha'^*} & q'_* q_* q'^* i_2^* p'^* p'_* p_* & \xRightarrow{\epsilon_{p'}} & q'_* q_* q'^* i_2^* p_* \\
 & & \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* \\
 & & q'_* q_* i_1^* p'^* p'_* p_* & \xRightarrow{\epsilon_{p'}} & q'_* q_* i_1^* p^* p_* & & \downarrow \epsilon_p \\
 & & & & & & q'_* q_* i_1^* \\
 & & & B_{D''} & & & \\
 & & & & & & B_D
 \end{array}$$

(ii) Similar to (i). \square

7.20. Fix a 2-faithful subcategory \mathcal{A} of \mathcal{C} and a pseudo-functor $A: \mathcal{A} \rightarrow (\mathcal{D}^{\text{op}})^{\text{adj}}$ with $P_1(A) = F$. We denote the pseudo-functor $L: \mathcal{A} \rightarrow \mathcal{D}$, composition of $P_2(A): \mathcal{A} \rightarrow (\mathcal{D}^{\text{op}})^{\text{op}}$ and the isomorphism $(\mathcal{D}^{\text{op}})^{\text{op}} \rightarrow \mathcal{D}$, by

$$i \mapsto i_!, \quad \alpha \mapsto \alpha_!.$$

Let D be a square (7.18.1) in \mathcal{C} where i and j are 1-cells of \mathcal{A} . The base change map A_D is by definition the following 2-cell in \mathcal{D}

$$(7.20.1) \quad j_! q^* \xRightarrow{\eta_i} j_! q^* i^* i_! \xRightarrow{\alpha^*} j_! (iq)^* i_! \xRightarrow{\alpha^*} j_! (pj)^* i_! \xRightarrow{\epsilon_j} j_! j^* p^* i_! \xRightarrow{\epsilon_j} p^* i_!.$$

If p and q are also 1-cells of \mathcal{A} , then A_D is also the composition

$$j_! q^* \xRightarrow{\eta_p} p^* p_! j_! q^* \xRightarrow{\alpha_!} p^* (pj)_! q^* \xRightarrow{\alpha_!} p^* (iq)_! q^* \xRightarrow{\epsilon_q} p^* i_! q_! q^* \xRightarrow{\epsilon_q} p^* i_!.$$

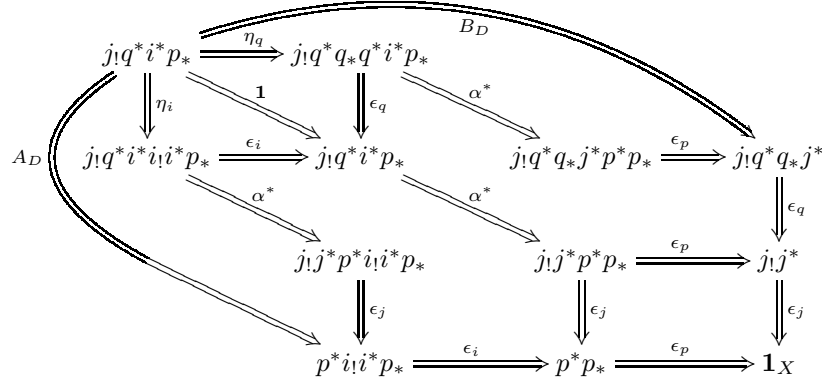
We have an analogue of 7.19 for A_D .

7.21. Let D be a square (7.18.1) in \mathcal{C} where i and j are 1-cells of \mathcal{A} , p and q are 1-cells of \mathcal{B} . Then

$$(B_D, A_D): (i^* p_*, p^* i_!, \eta_p \eta_i, \epsilon_p \epsilon_i) \Rightarrow (q_* j^*, j_! q^*, \eta_q \eta_j, \epsilon_q \epsilon_j)$$

is a 2-cell of \mathcal{D}^{adj} . In fact, the following diagrams commute

$$\begin{array}{ccccccc}
 1_Z & \xRightarrow{\eta_i} & i^* i_! & \xRightarrow{\eta_p} & i^* p_* p^* i_! & & \\
 \downarrow \eta_q & & \downarrow \eta_q & & \downarrow \eta_q & & \\
 q_* q^* & \xRightarrow{\eta_i} & q_* q^* i^* i_! & & q_* q^* i^* p_* p^* i_! & & \\
 \downarrow \eta_j & & \downarrow \eta_j & & \downarrow \eta_j & & \\
 q_* j^* j_! q^* & \xRightarrow{\eta_i} & q_* j^* j_! q^* i^* i_! & \xRightarrow{\alpha^*} & q_* j^* p^* i_! & \xRightarrow{\eta_p} & q_* j^* p^* p_* p^* i_! \\
 & \searrow \alpha^* & \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* \\
 & & q_* j^* j_! j^* p^* i_! & \xRightarrow{\epsilon_j} & q_* j^* p^* i_! & & \downarrow \epsilon_p \\
 & & & & & & q_* j^* p^* i_! \\
 & & & A_D & & & B_D
 \end{array}$$



It follows that B_D is invertible if and only if A_D is. In this case, the following diagram commutes

$$\begin{array}{ccccc}
 i!q_* & \xrightarrow{\eta_j} & i!q_*j^*j! & \xrightarrow{B_D^{-1}} & i!i^*p_*j! \\
 \downarrow \eta_p & & & & \downarrow \epsilon_i \\
 p_*p^*i!q_* & \xrightarrow{A_D^{-1}} & p_*j!q^*q_* & \xrightarrow{\epsilon_q} & p_*j!
 \end{array}$$

and we define $G_D: i!q_* \Rightarrow p_*j!$ to be the composition. In fact, the following diagram commutes

$$\begin{array}{ccccccc}
 & & i!q_* & \xrightarrow{\eta_j} & i!q_*j^*j! & & \\
 & \nearrow 1 & \uparrow \epsilon_q & & \downarrow B_D^{-1} & & \\
 i!q_* & \xrightarrow{\eta_q} & i!q_*q^*q_* & \xrightarrow{\eta_j} & i!q_*j^*j!q^*q_* & & \\
 \downarrow \eta_i & & & & \downarrow B_D^{-1} & & \\
 i!i^*i!q_* & \xrightarrow{\eta_p} & i!i^*p_*p^*i!q_* & \xrightarrow{A_D^{-1}} & i!i^*p_*j!q^*q_* & \xrightarrow{\epsilon_p} & i!i^*p_*j! \\
 \downarrow \epsilon_i & & & & \downarrow \epsilon_i & & \downarrow \epsilon_i \\
 i!q_* & \xrightarrow{\eta_p} & p_*p^*i!q_* & \xrightarrow{A_D^{-1}} & p_*j!q^*q_* & \xrightarrow{\epsilon_q} & p_*j!
 \end{array}$$

where the hexagon commutes because the following diagram commutes

$$\begin{array}{ccccc}
 & q_*q^* & \xrightarrow{\eta_j} & q_*j^*j!q^* & \\
 \nearrow \eta_q & & \searrow A_D & & \searrow B_D^{-1} \\
 1_Z & & q_*j^*p^*i! & & i^*p_*j!q^* \\
 \searrow \eta_i & & \searrow B_D^{-1} & & \nearrow A_D \\
 & i^*i! & \xrightarrow{\eta_p} & i^*p_*p^*i! &
 \end{array}$$

7.22. Let $f: X \rightarrow Y$ be a 1-cell of $\mathcal{A} \cap \mathcal{B}$. Then

$$(\epsilon_f^B, \eta_f^A): (f^*f_*, f^*f!, \eta_f^B \eta_f^A, \epsilon_f^B \epsilon_f^A) \Rightarrow (1_X, 1_X, 1_{1_X}, 1_{1_X})$$

is a 2-cell of \mathcal{D}^{adj} . It follows that ϵ_f^B is invertible if and only if η_f^A is. In this case, the following diagram commutes

$$\begin{array}{ccc}
 f! & \xrightarrow{\eta_f^B} & f_*f^*f! \\
 \downarrow (\epsilon_f^B)^{-1} & & \downarrow (\eta_f^A)^{-1} \\
 f!f^*f_* & \xrightarrow{\epsilon_f^A} & f_*
 \end{array}$$

and we define $\rho_f: f_! \Rightarrow f_*$ to be the composition. In fact, the following diagram commutes

$$\begin{array}{ccccc}
 & & f_* & & \\
 & \epsilon^A \nearrow & & \nwarrow \eta^A & \\
 f_! f^* f_* & \xrightarrow{\eta^A} & f_! f^* f_* f^* f_! & \xrightarrow{\epsilon^A} & f_* f^* f_! \\
 \downarrow \eta^B & & \downarrow & & \downarrow \eta^B \\
 f_* f^* f_! f^* f_* & \xrightarrow{\quad} & f_* f^* f_! f^* f_* f^* f_! & \xrightarrow{\quad} & f_* f^* f_* f^* f_! \\
 \downarrow \epsilon^B & & \downarrow & & \downarrow \epsilon^B \\
 f_* f^* f_! & \xrightarrow{\eta^A} & f_* f^* f_! f^* f_! & \xrightarrow{\epsilon^A} & f_* f^* f_! \\
 & \xrightarrow{1} & & & \\
 & & f_* f^* f_! & &
 \end{array}$$

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of composable 1-cells of $\mathcal{A} \cap \mathcal{B}$ with ϵ_f^B and ϵ_g^B invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
 & \rho_a & & & \\
 g_! f_! & \xrightarrow{\eta_f^B} & g_! f_* f^* f_! & \xrightarrow{(\eta_f^A)^{-1}} & g_! f_* \\
 \downarrow \eta_g^B & & \downarrow & & \downarrow \eta_g^B \\
 g_* g^* g_! f_* f^* f_! & \xrightarrow{(\eta_f^A)^{-1}} & g_* g^* g_! f_* & & \\
 \downarrow (\eta_g^A)^{-1} & & \downarrow & & \\
 & \rho_{gf} & & & g_* f_* \\
 & \rho_g & & &
 \end{array}$$

The following properties of G_D and ρ_f are similar to axioms (b), (b') of 7.3 and (c), (c') of 7.6.

Proposition 7.23. (i) In the situation of 7.19 (i), if the horizontal arrows are 1-cells of \mathcal{A} , the vertical arrows are 1-cells of \mathcal{B} , and if B_D and $B_{D'}$ are invertible, then the following diagram commutes

$$\begin{array}{ccccc}
 i_{3!} q'_* q_* & \xrightarrow{G_{D'}} & p_* i_{2!} q_* & \xrightarrow{G_D} & p'_* p_* i_{1!} \\
 \downarrow & & \downarrow & & \downarrow \\
 i_{3!} (q' q)_* & \xrightarrow{G_{D''}} & (p' p)_* i_{1!} & &
 \end{array}$$

(ii) In the situation of 7.19 (ii), if the horizontal arrows are 1-cells of \mathcal{A} , the vertical arrows are 1-cells of \mathcal{B} , and if B_D and $B_{D'}$ are invertible, then the following diagram commutes

$$\begin{array}{ccccc}
 i'_! i_! p_{1*} & \xrightarrow{G_D} & i'_! p_{2*} j_! & \xrightarrow{G_{D'}} & p_{3*} j'_! j_! \\
 \downarrow & & \downarrow & & \downarrow \\
 (i' i)_! p_{1*} & \xrightarrow{G_{D''}} & p_{3*} (j' j)_! & &
 \end{array}$$

(iii) Let D be a square (7.18.1) in \mathcal{C} where i and j are 1-cells of \mathcal{A} , p and q are 1-cells of $\mathcal{A} \cap \mathcal{B}$, ϵ_p^B , ϵ_q^B and B_D are invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
 i_! q_* & \xleftarrow{\rho_q} & i_! q_! & \xleftarrow{\quad} & (iq)_! \\
 \downarrow G_D & & \downarrow & & \downarrow \alpha_! \\
 p_* j_! & \xleftarrow{\rho_p} & p_! j_! & \xleftarrow{\quad} & (pj)_!
 \end{array}$$

(iv) Let D be a square (7.18.1) in \mathcal{C} where i and j are 1-cells of $\mathcal{A} \cap \mathcal{B}$, p and q are 1-cells of \mathcal{B} , ϵ_i^B , ϵ_j^B and B_D are invertible. Then the following diagram commutes

$$\begin{array}{ccccc} i!q_* & \xrightarrow{\rho_i} & i_*q_* & \xrightarrow{\quad} & (iq)_* \\ \Downarrow G_D & & & & \uparrow \alpha_* \\ p_*j! & \xrightarrow{\rho_j} & p_*j_* & \xrightarrow{\quad} & (pj)_* \end{array}$$

Proof. (i) Similar to (ii).

(ii) The following diagram commutes

$$\begin{array}{ccccccc} & & G_D & & & & \\ i'_1 i_1 p_{1*} & \xrightarrow{\eta_j} & i'_1 i_1 p_{1*} j^* j! & \xrightarrow{B_D^{-1}} & i'_1 i_1 i^* p_{2*} j! & \xrightarrow{\epsilon_i} & i'_1 p_{2*} j! \\ & & \downarrow \eta_{j'} & & \downarrow \eta_{j'} & & \downarrow \eta_{j'} \\ & & i'_1 i_1 p_{1*} j^* j'^* j'_1 j! & \xrightarrow{B_D^{-1}} & i'_1 i_1 i^* p_{2*} j'^* j'_1 j! & \xrightarrow{\epsilon_i} & i'_1 p_{2*} j'^* j'_1 j! \\ & & & & \downarrow B_{D'}^{-1} & & \downarrow B_{D'}^{-1} \\ & & & & i'_1 i_1 i^* i'^* p_{3*} j'_1 j_* & \xrightarrow{\epsilon_i} & i'_1 i'^* p_{3*} j'_1 j! \\ & & & & & & \downarrow \epsilon_{i'} \\ & & & & & & p_{3*} j'_1 j! \end{array}$$

$G_{D'} \curvearrowright$
 $G_{D''} \curvearrowleft$

(iii) Similar to (iv).

(iv) The following diagram commutes

$$\begin{array}{ccccccc} & & \rho_i & & & & \\ i!q_* & \xrightarrow{\eta_i^B} & i_* i^* i!q_* & \xrightarrow{(\eta_i^A)^{-1}} & i_* q_* & & \\ \downarrow \eta_j^A & & \downarrow & & \downarrow \eta_j^A \simeq & & \\ i!q_* j^* j! & \xrightarrow{\quad} & i_* i^* i!q_* j^* j! & \xrightarrow{\quad} & i_* q_* j^* j! & & \\ \downarrow B_D^{-1} & & \downarrow & & \downarrow B_D^{-1} \simeq & & \\ i!i^* p_* j! & \xrightarrow{\eta_i^B} & i_* i^* i!i^* p_* j! & \xrightarrow{(\eta_j^A)^{-1}} & i_* i^* p_* j! & & \\ \downarrow \epsilon_i^A & & \downarrow \epsilon_i^A & & \downarrow \alpha_* & & \\ p_* j! & \xrightarrow{\eta_i^B} & i_* i^* p_* j! & \xrightarrow{1} & p_* j_* j^* j! & \xrightarrow{(\eta_j^A)^{-1}} & p_* j_* \\ & & & & \uparrow \eta_j^B & & \\ & & & & \rho_j & & \end{array}$$

where the pentagon commutes because the following diagram commutes

$$\begin{array}{ccccccc} & & \eta_i^B & & \eta_q & & \\ p_* & \xrightarrow{\quad} & i_* i^* p_* & \xrightarrow{\quad} & i_* q_* q^* i^* p_* & & \\ \downarrow \eta_p & & \downarrow & & \downarrow \alpha^* & & \\ p_* p^* p_* & \xrightarrow{\eta_j^B} & p_* j_* j^* p^* p_* & \xrightarrow{\alpha^*} & i_* q_* j^* p^* p_* & & \\ \downarrow \epsilon_p & & \downarrow & & \downarrow \epsilon_p & & \\ p_* & \xrightarrow{\eta_j^B} & p_* j_* j^* & \xrightarrow{\alpha_*} & i_* q_* b^* & & \end{array}$$

$B_D \curvearrowright$
 $1_{p_*} \curvearrowleft$

□

Proposition 7.24. Let D be a square (7.18.1) in \mathcal{C} where i and j are 1-cells of $\mathcal{A} \cap \mathcal{B}$, ϵ_i^B , ϵ_j^B and α are invertible, and let D' be the square obtained by inverting α . Then the following diagram commutes

$$\begin{array}{ccc} j!q^* & \xrightarrow{A_D} & p^*i! \\ \downarrow \rho_j & & \downarrow \rho_i \\ j_*q^* & \xleftarrow{B_{D'}} & p^*i_* \end{array}$$

Proof. The following diagram commutes

$$\begin{array}{ccccc} & & \rho_j & & \\ & \curvearrowright & & \curvearrowright & \\ j!q^* & \xrightarrow{\eta_j^B} & j!j^*j_*q^* & \xrightarrow{(\eta_j^A)^{-1}} & j_*q^* \\ \downarrow \eta_i^A & & \downarrow \eta_i^A & & \downarrow \eta_i^A \\ j!q^*i^*i! & & j_*q^*i^*i! & & \\ \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* \\ j!j^*p^*i! & \xrightarrow{\eta_j^B} & j!j^*j_*p^*i! & \xrightarrow{(\eta_j^A)^{-1}} & j_*j^*p^*i! \\ \downarrow \epsilon_j^A & & \downarrow \epsilon_j^A & & \downarrow \epsilon_j^A \\ p^*i! & & j_*j^*p^*i! & & j_*j^*p^*i! \\ \downarrow \eta_i^B & & \downarrow \eta_i^B & & \downarrow \eta_i^B \\ p^*i_*i^*i! & \xrightarrow{\eta_j^B} & j_*j^*p^*i_*i^*i! & \xrightarrow{(\alpha^{-1})^*} & j_*q^*i^*i^*i! \\ \downarrow (\eta_i^A)^{-1} & & \downarrow (\eta_i^A)^{-1} & & \downarrow (\eta_i^A)^{-1} \\ p^*i_* & \xrightarrow{\eta_j^B} & j_*j^*p^*i_* & \xrightarrow{(\alpha^{-1})^*} & j_*q^*i^*i_* \\ & & B_{D'} & & \\ & \curvearrowleft & & \curvearrowleft & \end{array}$$

□

Proposition 7.25. Let

(7.25.1)

$$\begin{array}{ccccc} X' & \xrightarrow{j'} & Y' & & \\ \downarrow x & \searrow q' & \downarrow y & \searrow p' & \\ & Z' & \xrightarrow{i'} & W' & \\ \downarrow z & & \downarrow & & \downarrow w \\ X & \xrightarrow{j} & Y & & \\ \downarrow q & & \downarrow p & & \\ & Z & \xrightarrow{i} & W & \end{array}$$

be a cube in \mathcal{C} , where i, j, i', j' are 1-cells of \mathcal{A} , p, q, p', q' are 1-cells of \mathcal{B} , and the 2-cells of the right, left, front, back, bottom, top faces, I, I', J, J', K, K' , are respectively

$$py \Rightarrow wp', \quad qx \Rightarrow zq', \quad \beta: wi' \Rightarrow iz, \quad \beta': yj' \Rightarrow jx, \quad pj \Rightarrow iq, \quad p'j' \Rightarrow i'q'.$$

Assume that B_K and $B_{K'}$ are invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
 i'_! z^* q_* & \xrightarrow{B_{I'}} & i'_! q'_* x^* & \xrightarrow{G_{K'}} & p'_* j'_! x^* \\
 \downarrow A_J & & & & \downarrow A_{J'} \\
 w^* i_! q_* & \xrightarrow{G_K} & w^* p_* j_! & \xrightarrow{B_I} & p'_* y^* j_!
 \end{array}$$

Proof. The following diagram commutes

$$\begin{array}{ccccccc}
 & & & & A_J & & \\
 & & & & \nearrow & & \\
 i'_! z^* q_* & \xrightarrow{\eta_i} & i'_! z^* i_! q_* & \xrightarrow{\beta^*} & i'_! i'^* w^* i_! q_* & \xrightarrow{\epsilon_{i'}} & w^* i_! q_* \\
 \downarrow \eta_j & & \downarrow \eta_j & & \downarrow \eta_j & & \downarrow \eta_j \\
 i'_! q'_* q_* j^* j_! & \xrightarrow{\eta_i} & i'_! z^* i_! q_* j^* j_! & \xrightarrow{\beta^*} & i'_! i'^* w^* i_! q_* j^* j_! & \xrightarrow{\epsilon_{i'}} & w^* i_! q_* j^* j_! \\
 \downarrow B_{I'} & & \downarrow B_{I'} & & \downarrow B_K^{-1} & & \downarrow B_K^{-1} \\
 i'_! q'_* x^* & \xrightarrow{\eta_j} & i'_! q'_* x^* j^* j_! & & i'_! i'^* w^* i_! p_* j_! & & w^* i_! i^* p_* j_! \\
 \downarrow \eta_{j'} & & \downarrow \eta_{j'} & & \downarrow \epsilon_i & & \downarrow \epsilon_i \\
 i'_! q'_* j'^* j'_! x^* & & i'_! q'_* j'^* j'_! x^* j^* j_! & & i'_! i'^* w^* p_* j_! & \xrightarrow{\epsilon_{i'}} & w^* p_* j_! \\
 \downarrow B_{K'}^{-1} & & \downarrow B_{K'}^{-1} & & \downarrow B_I & & \downarrow B_I \\
 i'_! i'^* p'_* j'_! x^* & \xrightarrow{\eta_j} & i'_! i'^* p'_* j'_! x^* j^* j_! & \xrightarrow{\beta'^*} & i'_! i'^* p'_* j'_! j'^* y^* j_! & \xrightarrow{\epsilon_{j'}} & i'_! i'^* p'_* y^* j_! \\
 \downarrow \epsilon_{i'} & & \downarrow \epsilon_{i'} & & \downarrow \epsilon_{i'} & & \downarrow \epsilon_{i'} \\
 p'_* j'_! x^* & \xrightarrow{\eta_j} & p'_* j'_! x^* j^* j_! & \xrightarrow{\beta'^*} & p'_* j'_! j'^* y^* j_! & \xrightarrow{\epsilon_{j'}} & p'_* y^* j_! \\
 & & & & A_{J'} & &
 \end{array}$$

where the decagon is the outline of the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & q'_* x^* j^* & \xrightarrow{\eta_{j'}} & q'_* j'^* j'_! x^* j^* & \xrightarrow{B_{K'}^{-1}} & i'^* p'_* j'_! x^* j^* \\
 & & & & \downarrow \beta'^* & & \downarrow \beta'^* & & \downarrow \beta'^* \\
 & & & & q'_* j'^* y^* & \xrightarrow{\eta_{j'}} & q'_* j'^* j'_! j'^* y^* & \xrightarrow{B_{K'}^{-1}} & i'^* p'_* j'_! j'^* y^* \\
 & & & & \downarrow \epsilon_{j'} & & \downarrow \epsilon_{j'} & & \downarrow \epsilon_{j'} \\
 z^* q_* j^* & \xrightarrow{B_K^{-1}} & z^* i^* p_* & \xrightarrow{\eta_i} & z^* i^* p_* & \xrightarrow{\epsilon_i} & z^* i^* p_* & & z^* i^* p_* \\
 \downarrow \eta_i & & \downarrow \eta_i & & \downarrow \beta^* & & \downarrow \beta^* & & \downarrow \beta^* \\
 z^* i^* i_! q_* j^* & \xrightarrow{B_K^{-1}} & z^* i^* i_! i^* p_* & \xrightarrow{\epsilon_i} & z^* i^* p_* & & z^* i^* p_* & & z^* i^* p_* \\
 \downarrow \beta^* & & \downarrow \beta^* & & \downarrow \beta^* & & \downarrow \beta^* & & \downarrow \beta^* \\
 i'^* w^* i_! q_* j^* & \xrightarrow{B_K^{-1}} & i'^* w^* i_! i^* p_* & \xrightarrow{\epsilon_i} & i'^* w^* p_* & & i'^* w^* p_* & & i'^* w^* p_*
 \end{array}$$

where the octagon commutes by 7.19. □

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